UNIVERSITY OF DORTMUND

REIHE COMPUTATIONAL INTELLIGENCE

COLLABORATIVE RESEARCH CENTER 531

Design and Management of Complex Technical Processes and Systems by means of Computational Intelligence Methods

On Interpreting Fuzzy IF-THEN Rule Bases by Concepts of Functional Analysis

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No. CI-19/98

Technical Report ISSN 1433-3325 February 1998

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This work is a product of the Collaborative Research Center 531, “Computational Intelligence”, at the University of Dortmund and was printed with financial support of the Deutsche Forschungsgemeinschaft.

Abstract

Starting with ZADEH's fundamental paper "The Calculus of Fuzzy IF-THEN Rules" we mention five possible interpretations of a Fuzzy IF-THEN Rule Base. The interpretation of a given Fuzzy IF-THEN Rule Base strongly depends on the area where it is to be applied. In the paper presented we restrict this area to fuzzy control and approximate reasoning. Consequently, we interpret a Fuzzy IF-THEN Rule Base as a system of functional equations for determining a special functional operator. Then using the concepts of functional analysis and metric spaces we introduce the principles FATI and FITA and study their correctness and equivalence.

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Chapter 1

Introduction

In a "Special Lecture" held on the congress "Fuzzy Engineering toward Human Friendly Systems" (November 13–15, 1991, Yokohama, Japan) under the title

"The Calculus of Fuzzy IF-THEN Rules"

L. A. ZADEH developed a program for studying this field of problems [48, 49].

We put this program at the beginning of the report presented.

The principal questions addressed in the calculus of fuzzy IF-THEN rules are:

1. *What is the meaning of a fuzzy IF-THEN rule expressed as a joint or conditional possibility distribution?*
2. *What is the meaning of a collection of fuzzy IF-THEN rules?*
3. *How can blocks of fuzzy IF-THEN rules be combined?*
4. *How can a collection of fuzzy IF-THEN rules be interpolated?*
5. *How can algebraic operations on a collection of fuzzy IF-THEN rules be carried out?*
6. *How can fuzzy IF-THEN rules be inferred from observations?*
7. *How can fuzzy IF-THEN rules be compressed?*

With respect to the numerous applications of IF-THEN rules, for instance to fuzzy control, to fuzzy approximate reasoning, to fuzzy expert systems, to fuzzy pattern recognition and fuzzy clustering, to fuzzy decision making, etc., the importance of such investigations cannot be overestimated.

In the literature one can find numerous approaches and attempts to investigate this field of problems. Many of these are carried out only heuristically for special cases, others try to develop general and systematic concepts to study the area of IF-THEN rules (see References).

The starting point of our investigations is the concept of an IF-THEN rule base RB on a universe U.

In order to define this concept, by $\mathbb{R}$ and $\langle 0,1\rangle$ we denote the set of all real numbers r and the set of all real numbers r with $0 \leq r \leq 1$, respectively. For an arbitrary subset $S \subseteq \langle 0,1\rangle$ by $\operatorname{Sup} S$ we denote the supremum of S with respect to $\langle 0,1\rangle$ and the usual ordering $\leq$ of real numbers. Note that $\operatorname{Sup}\langle 0,1\rangle = 1$ and $\operatorname{Sup}\varnothing = 0$ where $\varnothing$ denotes the empty set.

A fuzzy set F on U is a mapping $F : U \to \langle 0,1\rangle$, i. e. we do not distinguish between a fuzzy set and its membership function because there is no reason for making such a distinction.

In the following we use the height $\mathrm{hgt}(F)$ of a fuzzy set F on U usually defined by

$$\mathrm{hgt}(F) =_{def} \mathrm{Sup}\,\{F(x) | x \in U\}\,.$$

Furthermore, if G is another fuzzy set on U we use the subset relation $F \subseteq G$ defined by

$$F \subseteq G =_{def} \forall x (x \in U \to F(x) \leq G(x)).$$

We recall that $\tau : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ is said to be a t-norm if and only if τ is monotone, commutative, associative, and fulfills the condition $\tau(r,1) = r$ for every $r \in \langle 0,1\rangle$.

For a fixed natural number $n \geq 1$, let $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$ be fuzzy sets on U.

Definition 1.1
A scheme of the form

$$RB: \quad \begin{array}{c} IF\, F_1\ THEN\, G_1 \\ \vdots \\ IF\, F_n\ THEN\, G_n \end{array}$$

is said to be a fuzzy IF-THEN rule base on U, shortly, an IF-THEN rule base.

Remark With respect to other more general definitions of an IF-THEN rule base we could say that *RB* is an IF-THEN rule base in *normal form*. We underline that IF-THEN rule bases in normal form are much more suitable for the following theoretical investigations while in applications more general forms are used, for instance, applying the *and* connective or other connectives in formulating the premises or the conclusions of IF-THEN rules. But we point out that the restriction to normal form does not mean a loss of generality because by suitable operations on fuzzy sets and by the construction principle of cylindrical extension one can transform an IF-THEN rule base used in applications into an IF-THEN rule base in normal form.

At first glance, an IF-THEN rule base in the sense defined above must be considered only as a syntactical object, i. e. without interpretation and without semantics.

We take the view that there are several possibilities to define an interpretation and to develop a semantics for an IF-THEN rule base *RB*. And, furthermore, which interpretation and semantics we introduce depends on the field where we want to apply the given IF-THEN rule base.

We can immediately see the following five areas where IF-THEN rule bases can be applied:

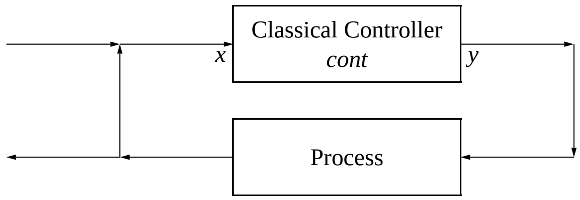
1. Fuzzy Control
2. Fuzzy Approximate Reasoning
3. Fuzzy Cluster Analysis
4. Fuzzy Decision Making
5. Fuzzy Computing.

Because of restricted space we cannot discuss all these aspects of applying IF-THEN rule bases and possible further aspects which we have not mentioned here.

In the following we shall discuss the field of Fuzzy Control and shall develop an interpretation and a semantics for IF-THEN rule bases with respect to their applications in this field.

We mention that this semantics can also be used for applying IF-THEN rule bases in the field of Fuzzy Approximate Reasoning.

In order to develop a semantics of an IF-THEN rule base for its application in Fuzzy Control we start with a (rough) description of a classical control circuit. It has the structure



We assume that the time is described by the set $\mathbb{R}$ of all real numbers and the starting point of our control circuit is 0. Therefore the input x and the output y of the controller are functions defined on the set $\langle 0, \infty)$ of all non-negative real numbers. For simplification we assume that for $t \in \langle 0, \infty)$ the values $x(t)$ and $y(t)$ are real numbers. (In general, one has to admit that $x(t)$ and $y(t)$ are elements from suitable linear spaces or even metric spaces.) The output function y is determined by a deterministic functional operator *cont* using the input function x as argument, i. e. $y = cont(x)$. The functional operator *cont* can be defined by methods of classical analysis as the example of the PID-controller shows, i. e. the function y is defined by x using the addition, multiplication, integration, and differentiation as follows

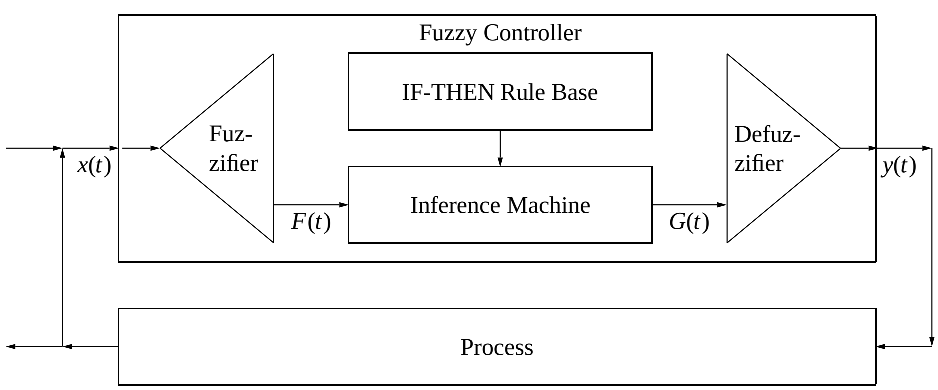
$$y(t) = a \cdot x(t) + b \cdot \int_0^t x(\tau) d\tau + c \cdot \frac{dx(\tau)}{d\tau}(t)$$

where $t \in \langle 0, \infty)$ and $a, b, c \in \mathbb{R}$. Obviously, this description of *cont* fails if the function x is not integrable or not differentiable. These difficulties will be overcome by fuzzy controllers as follows. For simplification we assume that the functional operator *cont* works “combinatorically”, i. e. that there exists a real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for every $t \in \langle 0, \infty)$ the equation

$$y(t) = f(x(t))$$

holds. More general cases, for example, taking into consideration the “past” $\{x(\tau) | 0 \leq \tau < t\}$ of $x(t)$ or the speed $\frac{dx}{dt}$ of the change of x, have hardly been considered and applied up to now in the field of fuzzy control. See, for instance, [12, 13, 35, 45].

For the following, we fix a time point $t \in \langle 0, \infty)$. Then the computation of the output $y(t)$ starting with the input $x(t)$ can be described by the following three steps.



Step 1. We have fixed a certain universe U and consider the set $FP(U)$ of all fuzzy sets on U. The block "Fuzzifier" defines a mapping φ which assigns to the real number $x(t) \in \mathbb{R}$ a fuzzy set $F(t) = \varphi(x(t))$ on U called the *fuzzification* of $x(t)$.

Step 2. The "Inference Machine" defines a mapping Φ which transforms the input fuzzy set $F(t)$ into an output fuzzy set $G(t) = \Phi(F(t))$. The Inference Machine uses the given IF-THEN rule base via a certain inference mechanism. To define the mapping Φ which is a functional operator $\Phi : FP(U) \to FP(U)$ will be the main problem of the report presented.

Step 3. The block "Defuzzifier" defines a mapping $\delta : FP(U) \to \mathbb{R}$. Using the concepts of functional analysis one can say that δ is a real functional defined on the set $FP(U)$.

Summarizing the three steps above one can state that the mapping f can be factorized in the form

$$f(x(t)) = \delta(\Phi(\varphi(x(t)))) = (\delta \circ \Phi \circ \varphi)(x(t)) \qquad (x(t) \in \mathbb{R}, t \in \langle 0, \infty)),$$

where $\quad \varphi : \mathbb{R} \to FP(U), \qquad \Phi : FP(U) \to FP(U), \qquad \delta : FP(U) \to \mathbb{R}.$

Now, for the following investigations we formulate our

Working Hypothesis

For studying classical control circuits, classical analysis, in particular the theory of differential equations including classical numerical mathematics, is a well-tried useful semantic instrument.

But for studying fuzzy control circuits this instrument fails, in general. Starting with the discussed factorization we think that modern functional analysis is a suitable mathematical framework and useful apparatus in order to investigate fuzzy control circuits.

Because of space restrictions we cannot investigate all of the mappings φ, Φ and δ, so in the following we shall study only the functional operator Φ. For investigations of the mappings φ and δ see [5, 25, 29] and [4, 21, 24, 31–34, 43, 44], respectively, for instance. Further references regarding defuzzification procedures can be found in e. g. [21, 44].

In the spirit of the fuzzy control approach described we interpret a given IF-THEN rule base

$$RB : \qquad \begin{array}{c} IF\, F_1\, THEN\, G_1 \\ \vdots \\ IF\, F_n\, THEN\, G_n \end{array}$$

as a "*partial definition*" of a functional operator $\Phi : FP(U) \to FP(U)$ fulfilling the functional equations

$$\begin{array}{c} \Phi(F_1) = G_1 \\ \vdots \\ \Phi(F_n) = G_n. \end{array} \tag{1}$$

In other words one can interpret RB as a system of functional equations for the unknown functional operator Φ.

To this approach we have to add two important remarks.

Remark 1. Up to now in publications one can find the following approach:

Let S be a binary fuzzy relation on U, i. e. $S : U \times U \to \langle 0,1\rangle$. Then for an arbitrary fuzzy set $F : U \to \langle 0,1\rangle$ on U a product $F \circ S$ is defined such that $F \circ S$ is a fuzzy set on U.

Very often the so-called "standard" product is used, defined by

$$(F \circ S)(y) =_{def} \operatorname{Sup}\{\min(F(x), S(x,y)) | x \in U\}.$$

Now, we know the interpretation of an IF-THEN rule base RB as a system of relational equations, i. e. as the problem of finding a binary fuzzy relation S such that all the equations

$$F_1 \circ S = G_1, \ldots, F_n \circ S = G_n$$

are satisfied.

Then using S the operator Φ is defined by

$$\Phi(F) =_{def} F \circ S$$

for every $F \in FP(U)$.

As we shall see later in section 3 the principle "FATI" can be subordinated to the "relational" approach described above whereas the principle "FITA" is more general and requires the investigation of systems of functional equations.

Remark 2. The system (1) of functional equations has a lot of solutions Φ, in general. Therefore restricting principles are necessary.

One of the most important restricting principles is the following: If we have a solution Φ of (1) then we want to have: If $F, F' : U \to \langle 0,1\rangle$ and F is similar to F' then $\Phi(F)$ is similar to $\Phi(F')$.

This heuristic formulation leads to the continuity of functional operators. Hence, in order to make this formulation more precise we need a topology in $FP(U)$. Assume we generate the considered topology in $FP(U)$ by a function $\rho : FP(U) \times FP(U) \to \langle 0,1\rangle$ where ρ is restricted as follows (see [3, 41]):

Definition 1.2

1. *ρ is said to be a (fuzzy)* ***co-tolerance*** *relation on $FP(U)$*
   $=_{def}$ *1.1. For every $F : U \to \langle 0,1\rangle$, $\rho(F,F) = 0$ (Co-Reflexivity).*

   1.2. For every $F, G : U \to \langle 0,1\rangle$, $\rho(F,G) = \rho(G,F)$ (Symmetry).

2. *ρ is said to be a* ***semi metric*** *on $FP(U)$*
   $=_{def}$ *ρ satisfies 1.1 and 1.2 and*

   2.1. For every $F, G, H : U \to \langle 0,1\rangle$, $\rho(F,H) \leq \rho(F,G) + \rho(G,H)$ (Triangle Inequality).

3. *ρ is said to be a* ***metric*** *on $FP(U)$*
   $=_{def}$ *ρ satisfies 1.1, 1.2, 2.1 and*

   3.1. For every $F, G : U \to \langle 0,1\rangle$, if $\rho(F,G) = 0$ then $F = G$.

For more information concerning metrics on $FP(U)$ see [3], for instance.

Now, using a co-tolerance relation ρ on $FP(U)$ we define for $\mathcal{F} \subseteq FP(U)$, $F \in \mathcal{F}$ and $\Phi : \mathcal{F} \to FP(U)$:

Definition 1.3

1. *Φ is said to be continuous in F with respect to ρ and $\mathfrak{F}$*
   $=_{def}$ *For every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $G \in \mathfrak{F}$, if $\rho(F,G) < \delta$ then $\rho(\Phi(F),\Phi(G)) < \varepsilon$.*

2. *Φ is said to be continuous in $\mathfrak{F}$ with respect to ρ*
   $=_{def}$ *For every $F \in \mathfrak{F}$, Φ is continuous in F with respect to ρ and $\mathfrak{F}$.*

Example 1.1 For arbitrary $F,G : U \to \langle 0,1\rangle$ we define

$$\rho_{\mathrm{C}}(F,G) =_{def} \mathrm{Sup}\,\{|F(x)-G(x)| \,|\, x \in U\}.$$

Hence, ρ_{C} is a metric on $FP(U)$. It is called CHEBYSHEV metric.

Example 1.2 Assume U is not empty but finite, p is a fixed real number with $p > 0$, $F,G : U \to \langle 0,1\rangle$.

$$\rho_p(F,G) =_{def} \left[\sum_{x\in U} |F(x)-G(x)|^p\right]^{\frac{1}{p}}.$$

Obviously, ρ_p is a metric on $FP(U)$. ρ_1 is called HAMMING-distance and ρ_2 is the well-known EUCLIDian distance. Finally, the CHEBYSHEV metric can be generated by $\lim\limits_{p\to\infty} \rho_p(F,G)$.

Chapter 2

The Special Case of One IF-THEN Rule

2.1 The Compositional Rule of Inference

Let F and G be fixed fuzzy sets on U, i. e. $F, G : U \to \langle 0,1\rangle$. We consider the IF-THEN rule $R =_{def} IF F\ THEN G$. In section 1 we have stated that with respect to fuzzy control an IF-THEN rule defines a functional equation, i. e. we have to construct a functional operator $\Phi^R : FP(U) \to FP(U)$ such that the equation $\Phi^R(F) = G$ holds where the choice of Φ^R is restricted by additional conditions, for instance by a certain version of continuity.

A fundamental approach to construct such an operator is the *Compositional Rule of Inference* introduced by L. A. ZADEH in [47] which can be described as follows:

Definition 2.1.1 (Compositional Rule of Inference)

1. *We define a binary fuzzy relation S on U by*

$$S(x,y) =_{def} \min(F(x), G(y)) \qquad (x,y \in U).$$

2. *For an arbitrary $F' : U \to \langle 0,1\rangle$ we define the inferred set $G' : U \to \langle 0,1\rangle$ by*

$$G'(y) =_{def} \mathrm{Sup}\{\min(F'(x), S(x,y)) | x \in U\} \qquad (y \in U).$$

3. $\Phi^R(F') =_{def} G'$.

The compositional rule of inference can be motivated and justified by the following mapping concept of crisp set theory.

Assume $A \subseteq U$ and $T \subseteq U \times U$.

Definition 2.1.2
The T-image $A \circ T$ of the set A is defined as $A \circ T =_{def} \{y | \exists x(x \in A \wedge [x,y] \in T)\}$.

Let us recall that for $F : U \to \langle 0,1\rangle$ and $S : U \times U \to \langle 0,1\rangle$ the kernels $\mathrm{Ker}(F)$ and $\mathrm{Ker}(S)$ are defined by

$$\mathrm{Ker}(F) =_{def} \{x | x \in U \wedge F(x) = 1\}$$
$$\mathrm{Ker}(S) =_{def} \{[x,y] | x,y \in U \wedge S(x,y) = 1\}.$$

Then the compositional rule of inference is justified by the following compatibility theorem:

Theorem 2.1.1

If 1. $G'(y) = \mathrm{Sup}\{\min(F'(x), S(x,y)) \mid x \in U\}$ *for every* $y \in U$

2. $\forall y (y \in U \rightarrow \exists x (x \in U \wedge G'(y) = \min(F'(x), S(x,y))))$

then $\mathrm{Ker}(G') = \mathrm{Ker}(F') \circ \mathrm{Ker}(S)$.

Proof By definition of Ker we have to show

(1) $$y \in \mathrm{Ker}(G') \leftrightarrow y \in (\mathrm{Ker}(F') \circ \mathrm{Ker}(S)).$$

By definition of $\circ$ we have

(2) $$y \in \mathrm{Ker}(F') \circ \mathrm{Ker}(S) \leftrightarrow \exists x (x \in \mathrm{Ker}(F') \wedge [x,y] \in \mathrm{Ker}(S)),$$

hence by definition of Ker it is sufficient to show

(3) $$G'(y) = 1 \leftrightarrow \exists x (F'(x) = 1 \wedge S(x,y) = 1)$$

I. ($\rightarrow$)

Assume

(4) $$G'(y) = 1,$$

hence by definition of G'

(5) $$\mathrm{Sup}\{\min(F'(x), S(x,y)) \mid x \in U\} = 1.$$

Because of assumption 2 there exists an $x \in U$ such that

(6) $$\min(F'(x), S(x,y)) = 1,$$

hence

(7) $$F'(x) = 1 \quad \text{and} \quad S(x,y) = 1.$$

II. ($\leftarrow$)

Assume there exists an $x \in U$ such that

(8) $$F'(x) = 1 \quad \text{and} \quad S(x,y) = 1,$$

hence

(9) $$\min(F'(x), S(x,y)) = 1,$$

hence

(10) $$\mathrm{Sup}\{\min(F'(x), S(x,y)) \mid x \in U\} = 1,$$

i. e. $G'(y) = 1$. ■

Finally, we interpret the compositional rule of inference as a procedure for defining a functional operator Φ^R by

$$\Phi^R(F')(y) =_{def} \mathrm{Sup}\{\min(F'(x), \min(F(x), G(y))) \mid x \in U\} \qquad (y \in U)$$

for a given (fixed) IF-THEN rule $R: IF\,F\ THEN\,G$ and for arbitrary $F': U \rightarrow \langle 0,1 \rangle$.

2.2 Fuzzy Approximate Reasoning

Now, we discuss a second application of the Compositional Rule of Inference.

In the framework of Fuzzy Approximate Reasoning the so-called Generalized Modus Ponens is introduced as a "fuzzy deduction scheme" of the following form

$$\begin{array}{ll} \text{Rule:} & \textit{IF } F \ \textit{ THEN } G \\ \text{Premise:} & \quad F' \\ \hline \text{Conclusion:} & \qquad\qquad G' \end{array}$$

where F, F', G, G' are fuzzy sets on U.

Obviously, the Generalized Modus Ponens coincides with the usual Modus Ponens if we assume $F' = F$ and $G' = G$, i. e.

$$\begin{array}{l} \textit{IF } F \textit{ THEN } G \\ \quad F \\ \hline \qquad\qquad G \end{array}.$$

But the use of the Generalized Modus Ponens differs from the (usual) Modus Ponens with respect to some essential features.

The application of the Modus Ponens is very clear. Assume that $\mathcal{L}$ is a logical system with a certain concept of "theorem" where theorems are defined "model-based" or "rule-based", for instance.

The use of the Modus Ponens is correct (and permitted) if the following justification (or soundness) lemma holds in $\mathcal{L}$: *If "IF F THEN G" and "F" are theorems then "G" is a theorem.* The conclusion G can be obtained by the trivial syntactic operation of detaching the premise "F" of the implication "*IF F THEN G*".

The application of the Generalized Modus Ponens is much more complicated and is not comparable with the application of the (usual) Modus Ponens. The reason is that the Generalized Modus Ponens cannot be used as a rule of detachment in the sense of a logical system with a certain concept of "theorem".

A semantics for a sensible application of the Generalized Modus Ponens can be developed by the following two steps.

Step 1 Assume that the fuzzy sets F and G on U are fuzzy descriptions of states of a given system. Using the well-known "tomato example" we put

$$F =_{def} \text{ the tomato } t \text{ is red}$$
$$G =_{def} \text{ the tomato } t \text{ is ripe.}$$

Then we interpret the expression

$$\textit{IF } F \textit{ THEN } G,$$

i. e.

$$\textit{IF} \text{ the tomato } t \text{ is red } \textit{THEN} \text{ the tomato } t \text{ is ripe,}$$

as a description of the fact that G is a logical (or, possibly, a causal) consequence of F.

On the basis of this interpretation we call a functional operator $\Phi : FP(U) \to FP(U)$ which fulfills the equation

$$\Phi(F) = G$$

an interpretation of the expression *IF F THEN G*.

Step 2 Following the "philosophy" of the Compositional Rule of Inference for an arbitrary fuzzy set F' on U (interpreted as a "generalized" premise) we define the conclusion G' by

$$G' =_{def} \Phi(F').$$

So, we can say that using a (fixed) interpretation Φ of the expression *IF F THEN G* we have defined the (semantic) function of the Generalized Modus Ponens.

Remark The proposed concept of interpretation Φ is very general. Using our tomato example we derive the monotony as a further condition which must be fulfilled by an interpretation of *IF F THEN G*.

Definition 2.2.1
Φ *is said to be monotone*
$=_{def} \forall H \forall H' (H, H' \in FP(U) \wedge H \subseteq H' \rightarrow \Phi(H) \subseteq \Phi(H'))$

We put

$$F' =_{def} \text{ the tomato } t \text{ is dark red}$$
$$G' =_{def} \text{ the tomato } t \text{ is very ripe.}$$

Then interpreting the words "dark red" and "very ripe" we have $F' \subseteq F$ and $G' \subseteq G$.

This holds on the basis of the following statement:

If Φ is an interpretation of *IF F THEN G*,
Φ is monotone, and
$F' \subseteq F$
then $G' \subseteq G$.

Proof From

$$F' \subseteq F$$

by monotonicity of Φ we get

(1) $$\Phi(F') \subseteq \Phi(F).$$

Because Φ is an interpretation of *IF F THEN G*, we get

(2) $$\Phi(F) = G.$$

By definition we have

(3) $$G' =_{def} \Phi(F').$$

But (1), (2), and (3) imply

(4) $$G' \subseteq G.$$

■

2.3 Interpretation of Fuzzy IF-THEN Rules

We underline that the construction of a functional operator with the properties described above is a new problem.

To illustrate this problem we consider the following "classical" construction (see definition 2.1.1)

$$\Phi^R(F')(y) =_{def} \operatorname{Sup}\{\min(F'(x), \min(F(x), G(y))) | x \in U\}$$

where $R =_{def}$ $IF\, F\; THEN\, G$.

Obviously, Φ^R is a monotone functional operator because min and Sup are monotone.

The equation $\Phi^R(F) = G$ only holds if F and G fulfill certain suppositions (see corollary 2.4.3).

We have to state that by the compositional rule of inference defined in definition 2.1.1 we cannot construct enough solutions which are necessary in applications and for developing a good and rich theory. Therefore we generalize the concept of interpretation and the concept of the compositional rule of inference as follows:

Definition 2.3.1
$\mathfrak{J} = [\pi, \kappa, Q]$ *is said to be an* ***interpretation*** *(of a single IF-THEN rule)*
$=_{def}$
1. *$\pi, \kappa : \langle 0,1\rangle^2 \to \langle 0,1\rangle$, i. e. π and κ are binary real functions from $\langle 0,1\rangle^2$ into $\langle 0,1\rangle$.*
2. *$Q : \mathfrak{P}\langle 0,1\rangle \to \langle 0,1\rangle$, i. e. Q is a mapping from the power set $\mathfrak{P}\langle 0,1\rangle$ of the closed unit interval $\langle 0,1\rangle$ into this interval; Q is also called real quantifier.*

Definition 2.3.2 (Generalized Compositional Rule of Inference)
1. *The function π, called "implication function", interprets the rule $R = IF\, F\; THEN\, G$ by defining the binary fuzzy "implication relation" S on U as*

$$S(x,y) =_{def} \pi(F(x), G(y)) \qquad (x, y \in U).$$

2. *For an arbitrary $F' : U \to \langle 0,1\rangle$ the image $G' : U \to \langle 0,1\rangle$ is inferred using S, κ and Q as follows:*

$$G'(y) =_{def} Q(\{\kappa(F'(x), S(x,y)) | x \in U\}) \qquad (y \in U).$$

3. *Analogous to the definition of the operator Φ^R (see definition 2.1.1) on the basis of the given interpretation $\mathfrak{J} = [\pi, \kappa, Q]$ we define*

$$\Phi^R_{\mathfrak{J}}(F')(y) =_{def} Q(\{\kappa(F'(x), \pi(F(x), G(y))) | x \in U\}) \qquad (y \in U)$$

   for every $F' : U \to \langle 0,1\rangle$ in coincidence with point 2 above.

2.4 Concepts of Correctness for Interpreted Fuzzy IF-THEN Rules. Criteria for Local Correctness

For the formulation of the following theorems of this chapter we need the following definition, where ρ is a co-tolerance relation on $FP(U)$, $\mathfrak{F} \subseteq FP(U)$ and $F \in \mathfrak{F}$.

Definition 2.4.1
1. *$\Phi^R_{\mathfrak{J}}$ is said to be* ***locally correct***
   $=_{def} \Phi^R_{\mathfrak{J}}(F) = G$

2. $\Phi_{\mathfrak{J}}^{R}$ *is said to be **weakly globally correct** with respect to* ρ *and* $\mathcal{F}$
   $=_{def}$ $\Phi_{\mathfrak{J}}^{R}$ *is locally correct and* $\Phi_{\mathfrak{J}}^{R}$ *is continuous in* F *with respect to* ρ *and* $\mathcal{F}$.
3. $\Phi_{\mathfrak{J}}^{R}$ *is said to be **globally correct** with respect to* ρ *and* $\mathcal{F}$
   $=_{def}$ $\Phi_{\mathfrak{J}}^{R}$ *is locally correct and* $\Phi_{\mathfrak{J}}^{R}$ *is continuous in all "points"* $F \in \mathcal{F}$ *with respect to* ρ *and* $\mathcal{F}$.

Theorem 2.4.1
If $R = IF\,F\ THEN\,G$ *and* $\mathfrak{J} = [\pi, \kappa, Q]$ *is an interpretation of* R *such that*

1. κ *and* π *are t-norms*
2. $Q = \mathrm{Sup}$

then for every $y \in U$, $\quad \Phi_{\mathfrak{J}}^{R}(F)(y) \le G(y)$.

Proof
Because π and κ are t-norms, we have

(1) $\quad$ for every $r,s \in \langle 0,1\rangle$, $\pi(r,s) \le \min(r,s)$ and $\kappa(r,s) \le \min(r,s)$,

hence

(2) $\quad$ for every $x,y \in U$, $\pi(F(x),G(y)) \le \min(F(x),G(y))$,

From (1) and (2) for every $x,y \in U$, we obtain

(3)
$$\begin{aligned}
&\kappa(F(x),\pi(F(x),G(y)))\\
&\le \min(F(x),\pi(F(x),G(y)))\\
&\le \min(F(x),\min(F(x),G(y))) \quad \text{because min is monotone}\\
&= \min(F(x),G(y))\\
&\le G(y),
\end{aligned}$$

hence

(4)
$$\Phi_{\mathfrak{J}}^{R}(F)(y) =_{def} \mathrm{Sup}\{\kappa(F(x),\pi(F(x),G(y))) \mid x \in U\} \le G(y).$$

■

Remark For the validity of theorem 2.4.1 we do not need the complete assumption 1, i. e. that π and κ are t-norms. As the above proof shows the condition (1) is sufficient.

Theorem 2.4.2
If $R = IF\,F\ THEN\,G$ *and* $\mathfrak{J} = [\pi, \kappa, Q]$ *is an interpretation of* R *such that*

1. κ *and* π *are t-norms*
2. $Q = \mathrm{Sup}$
3. *At least one of the following three conditions is satisfied*
   3.1. $\mathrm{hgt}(F) \ge \mathrm{hgt}(G)$ *and* $\pi = \kappa = \min$ *or*
   3.2. $\mathrm{hgt}(F) = 1$ *and* π, κ *are continuous* *or*
   3.3. *there is an* $x \in U$ *such that* $F(x) = 1$

then for every $y \in U$, $\quad G(y) \le \Phi_{\mathfrak{J}}^{R}(F)(y)$.

Proof
Case 3.1
By assumption we have

$$\mathrm{hgt}(F) \geq \mathrm{hgt}(G) \text{ and } \pi = \kappa = \min.$$

The operator $\Phi_{\mathfrak{J}}^{R}$ is defined by

(1) $$\Phi_{\mathfrak{J}}^{R}(F)(y) =_{def} \mathrm{Sup}\,\{\min(F(x), \min(F(x), G(y))) | \, x \in U\},$$

hence by

$$\forall r \forall s \big(r, s \in \langle 0, 1\rangle \rightarrow \min(r, \min(r, s)) = \min(r, s)\big),$$

(2) $$\Phi_{\mathfrak{J}}^{R}(F)(y) = \mathrm{Sup}\,\{\min(F(x), G(y)) | \, x \in U\}.$$

Because min is continuous, we get

(3) $$\mathrm{Sup}\,\{\min(F(x), G(y)) | \, x \in U\} \geq \min(\mathrm{Sup}\,\{F(x) | x \in U\}, G(y)).$$

From the assumption $\mathrm{hgt}(F) \geq \mathrm{hgt}(G)$ we get

(4) $$\forall y (y \in U \rightarrow \mathrm{Sup}\,\{F(x) | x \in U\} \geq G(y)),$$

hence

(5) $$\forall y (y \in U \rightarrow \min(\mathrm{Sup}\,\{F(x) | x \in U\}, G(y)) = G(y)),$$

hence by (2), (3), and (5) we get

$$\Phi_{\mathfrak{J}}^{R}(F)(y) \geq G(y).$$

Case 3.2
By assumption we have that

$$\mathrm{hgt}(F) = 1 \text{ and } \pi, \kappa \text{ are continuous.}$$

Because π and κ are continuous

(6) the function φ defined by $\varphi(r, s) =_{def} \kappa(r, \pi(r, s))$ $(r, s \in \langle 0, 1\rangle)$ is also continuous.

From (6) we obtain

(7) $$\begin{aligned} &\mathrm{Sup}\,\{\kappa(F(x), \pi(F(x), G(y))) | \, x \in U\} \\ &\geq \kappa(\mathrm{Sup}\,\{F(x) | x \in U\}, \pi(\mathrm{Sup}\,\{F(x) | x \in U\}, G(y))) \,. \end{aligned}$$

Because of the definition

$$\mathrm{hgt}(F) =_{def} \mathrm{Sup}\,\{F(x) | x \in U\}$$

and the assumption

$$\mathrm{hgt}(F) = 1$$

and

$$\kappa \text{ and } \pi \text{ are t-norms,}$$

we obtain

$$
(8)\qquad
\begin{aligned}
&\kappa(\mathrm{Sup}\{F(x)|x\in U\},\pi(\mathrm{Sup}\{F(x)|x\in U\},G(y)))\\
&=\kappa(\mathrm{hgt}(F),\pi(\mathrm{hgt}(F),G(y)))\\
&=\kappa(1,\pi(1,G(y)))\\
&=\kappa(1,G(y))\\
&=G(y),
\end{aligned}
$$

hence by (7) and definition of $\Phi_{\mathfrak{J}}^{R}(F)(y)$

$$
(9)\qquad \Phi_{\mathfrak{J}}^{R}(F)(y)\geq G(y).
$$

Case 3.3
By assumption there exists an $x_0\in U$ such that

$$
F(x_0)=1\,.
$$

By definition of $\Phi_{\mathfrak{J}}^{R}$ we have to show

$$
(10)\qquad \mathrm{Sup}\{\kappa(F(x),\pi(F(x),G(y)))|\,x\in U\}\geq G(y),
$$

hence it is sufficient to prove

$$
(11)\qquad \exists x(x\in U\wedge\kappa(F(x),\pi(F(x),G(y)))\geq G(y)).
$$

We choose $x=_{def}x_0$.
Because π and κ are t-norms we obtain

$$
(12)\qquad
\begin{aligned}
&\kappa(F(x_0),\pi(F(x_0),G(y)))\\
&=\kappa(1,\pi(1,G(y)))\\
&=\kappa(1,G(y))\\
&=G(y).
\end{aligned}
$$

■

Corollary 2.4.3
If $R = \text{IF}\,F\text{ THEN}\,G$ and $\mathfrak{J}=[\pi,\kappa,Q]$ is an interpretation of R such that

1. *κ and π are t-norms*
2. *$Q=\mathrm{Sup}$*
3. *At least one of the following three conditions is satisfied*
   - *3.1.* $\mathrm{hgt}(F)\geq\mathrm{hgt}(G)$ *and* $\pi=\kappa=\min$ *or*
   - *3.2.* $\mathrm{hgt}(F)=1$ *and π,κ are continuous or*
   - *3.3. there exists an $x\in U$ such that $F(x)=1$*

then $\Phi_{\mathfrak{J}}^{R}(F)=G$, i. e. $\Phi_{\mathfrak{J}}^{R}$ is locally correct.

Proof By applying theorem 2.4.1 and 2.4.2. ■

By analyzing the proofs of theorem 2.4.1 and 2.4.2 we realize that both theorems are valid under essentially weaker assumptions.

In particular, we shall see that the concept of norm is irrelevant because the above mentioned theorems hold without using t-norms.

Theorem 2.4.4
If R = IF F THEN G and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. $\forall r \forall s \big(r,s \in \langle 0,1 \rangle \to \pi(r,s) \le \min(r,s) \wedge \kappa(r,s) \le \min(r,s) \big)$
2. $Q = \mathrm{Sup}$

then for every $y \in U$,

$$\Phi_{\mathfrak{J}}^{R}(F)(y) \le G(y).$$

Proof See the proof of theorem 2.4.1. It is not necessary that κ and π are t-norms. ■

Another generalization of theorem 2.4.1 is the following

Theorem 2.4.5
If R = IF F THEN G and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. *π and κ satisfy the "boundary condition"*

$$\forall s \big(s \in \langle 0,1 \rangle \to \pi(1,s) \le s \wedge \kappa(1,s) \le s \big)$$

2. *π and κ are monotone with respect to their first arguments, i. e.*

$$\forall r \forall r' \forall t \big(r, r', t \in \langle 0,1 \rangle \wedge r \le r' \to \pi(r,t) \le \pi(r',t) \wedge \kappa(r,t) \le \kappa(r',t) \big)$$

3. $Q = \mathrm{Sup}$

then for every $y \in U$,

$$\Phi_{\mathfrak{J}}^{R}(F)(y) \le G(y).$$

Proof By definition of $\Phi_{\mathfrak{J}}^{R}$ we have to show

(1) $$\Phi_{\mathfrak{J}}^{R}(F)(y) =_{def} \mathrm{Sup}\{ \kappa(F(x), \pi(F(x), G(y))) | x \in U \} \le G(y),$$

hence it is sufficient to prove

(2) $$\forall x (x \in U \to \kappa(F(x), \pi(F(x), G(y))) \le G(y)).$$

Now by assumption 1 and the monotonicity of κ in its first argument, we obtain

(3) $$\kappa(F(x), \pi(F(x), G(y))) \le \kappa(1, \pi(F(x), G(y))) \le \pi(F(x), G(y)),$$

furthermore, by assumption 1 and the monotonicity of π in its first argument, we obtain

(4) $$\pi(F(x), G(y)) \le \pi(1, G(y)) \le G(y),$$

hence (3) and (4) imply (2). ■

Now, we generalize theorem 2.4.2.

Theorem 2.4.6
If R = IF F THEN G and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. $Q = \mathrm{Sup}$
2. *At least one of the following three conditions is satisfied*

2.1. $\operatorname{hgt}(F) \geq \operatorname{hgt}(G)$ *and*
$\forall r \forall s \big(r, s \in \langle 0,1\rangle \rightarrow \min(r,s) \leq \pi(r,s) \wedge \min(r,s) \leq \kappa(r,s)\big)$

2.2. $\operatorname{hgt}(F) = 1$, *$\pi$ is continuous in its first argument, κ is continuous, and* $\forall s \big(s \in \langle 0,1\rangle \rightarrow \pi(1,s) = \kappa(1,s) = s\big)$

2.3. $\exists x (x \in U \wedge F(x) = 1)$ *and* $\forall s \big(s \in \langle 0,1\rangle \rightarrow \pi(1,s) = \kappa(1,s) = s\big)$

then for every $y \in U$,

$$G(y) \leq \Phi_{\mathfrak{J}}^{R}(F)(y).$$

Proof See the proof of theorem 2.4.2. ■

With respect to assumption 2.1 of theorem 2.4.6 we can modify this theorem as follows.

Theorem 2.4.7
If $R = \mathit{IF}\, F\, \mathit{THEN}\, G$ and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. $\operatorname{hgt}(F) \geq \operatorname{hgt}(G)$
2. $\forall r \forall s \big(r, s \in \langle 0,1\rangle \rightarrow \pi(r,s) \geq s \wedge \kappa(r,s) \geq s\big)$
3. *κ is monotone in its first argument*
4. *the function φ defined by*

$$\varphi(r,s) =_{def} \kappa(r, \pi(r,s)) \quad (r, s \in \langle 0,1\rangle)$$

is continuous in its first argument

5. $Q = \operatorname{Sup}$

then for every $y \in U$,

$$G(y) \leq \Phi_{\mathfrak{J}}^{R}(F)(y).$$

Proof From the assumption

$$\operatorname{hgt}(F) \geq \operatorname{hgt}(G)$$

we get for every $y \in U$,

$$\operatorname{hgt}(F) \geq G(y), \tag{1}$$

hence by assumption 2 for π

$$\pi(\operatorname{hgt}(F), G(y)) \geq G(y), \tag{2}$$

hence by assumption 3

$$\kappa(\operatorname{hgt}(F), \pi(\operatorname{hgt}(F), G(y))) \geq \kappa(\operatorname{hgt}(F), G(y)), \tag{3}$$

hence by assumption 2 for κ

$$\kappa(\operatorname{hgt}(F), \pi(\operatorname{hgt}(F), G(y))) \geq G(y). \tag{4}$$

By assumption 4 we obtain

$$\begin{aligned} &\operatorname{Sup}\{\kappa(F(x), \pi(F(x), G(y))) \mid x \in U\} \\ &\geq \kappa(\operatorname{Sup}\{F(x) \mid x \in u\}, \pi(\operatorname{Sup}\{F(x) \mid x \in U\}, G(x))), \end{aligned} \tag{5}$$

consequently by (4) and the definition of $\Phi_{\mathfrak{J}}^{R}$ we obtain

$$G(y) \leq \Phi_{\mathfrak{J}}^{R}(F)(y). \tag{6}$$

■

Corollary 2.4.8
If $R = IF\,F\ THEN\,G$ *and* $\mathfrak{J} = [\pi, \kappa, Q]$ *is an interpretation of R such that*

1. $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \pi(r,s) \le \min(r,s) \wedge \kappa(r,s) \le \min(r,s)\big)$
2. $Q = \mathrm{Sup}$
3. *At least one of the following three conditions is satisfied*
   - *3.1.* $\mathrm{hgt}(F) \ge \mathrm{hgt}(G)$ *and* $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \min(r,s) \le \pi(r,s) \wedge \min(r,s) \le \kappa(r,s)\big)$
   - *3.2.* $\mathrm{hgt}(F) = 1$*,* π *is continuous in its first argument,* κ *is continuous, and* $\forall s \big(s \in \langle 0,1\rangle \to \pi(1,s) = \kappa(1,s) = s\big)$
   - *3.3.* $\exists x (x \in U \wedge F(x) = 1)$ *and* $\forall s \big(s \in \langle 0,1\rangle \to \pi(1,s) = \kappa(1,s) = s\big)$

then $\Phi^R_{\mathfrak{J}}(F) = G$*, i. e.* $\Phi^R_{\mathfrak{J}}$ *is locally correct.*

Proof See theorem 2.4.4 and 2.4.6. ■

Corollary 2.4.9
If $R = IF\,F\ THEN\,G$ *and* $\mathfrak{J} = [\pi, \kappa, Q]$ *is an interpretation of R such that*

1. $\forall s \big(s \in \langle 0,1\rangle \to \pi(1,s) = \kappa(1,s) = s\big)$
2. π *and* κ *are monotone in their first arguments*
3. $Q = \mathrm{Sup}$
4. *at least one of the following conditions is satisfied*
   - *4.1.* $\mathrm{hgt}(F) \ge \mathrm{hgt}(G)$ *and* $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \min(r,s) \le \pi(r,s) \wedge \min(r,s) \le \kappa(r,s)\big)$
   - *4.2.* $\mathrm{hgt}(F) = 1$*,* π *is continuous in its first argument, and* κ *is continuous*
   - *4.3.* $\exists x (x \in U \wedge F(x) = 1)$

then $\Phi^R_{\mathfrak{J}}(F) = G$*, i. e.* $\Phi^R_{\mathfrak{J}}$ *is locally correct.*

Proof Applying theorem 2.4.5 and 2.4.6 ■

Remark By combining

1. theorem 2.4.4 and 2.4.7 and
2. theorem 2.4.5 with 2.4.7

we obtain two further corollaries expressing the local correctness of $\Phi^R_{\mathfrak{J}}$ under sufficient conditions.

Formulating these corollaries is left to the reader.

Theorem 2.4.10
If $R = IF\,F\ THEN\,G$ *and* $\mathfrak{J} = [\pi, \kappa, Q]$ *is an interpretation of R such that*

1. π *is defined by*
$$\pi(r,s) =_{def} \mathrm{Sup}\left\{t \middle| t \in \langle 0,1\rangle \text{ and } \kappa(r,t) \le s\right\} \text{ for every } r, s \in \langle 0,1\rangle$$
2. κ *is continuous in its second argument*
3. $Q = \mathrm{Sup}$

then for every $y \in U$, $\quad \Phi_{\Im}^{R}(F)(y) \le G(y)$.

Proof By definition of π we have

(1) $$\pi(F(x), G(y)) =_{def} \mathrm{Sup}\left\{ t \middle| t \in \langle 0,1\rangle \text{ and } \kappa(F(x),t) \le G(y) \right\},$$

hence

(2) $$\begin{aligned}&\kappa(F(x), \pi(F(x), G(y)))\\ &= \kappa\left(F(x), \mathrm{Sup}\left\{ t \middle| t \in \langle 0,1\rangle \text{ and } \kappa(F(x),t) \le G(y) \right\}\right).\end{aligned}$$

Because κ is continuous with respect to its second argument we get

(3) $$\begin{aligned}&\kappa\left(F(x), \mathrm{Sup}\left\{ t \middle| t \in \langle 0,1\rangle \text{ and } \kappa(F(x),t) \le G(y) \right\}\right)\\ &\le \mathrm{Sup}\left\{ \kappa(F(x),t) \middle| t \in \langle 0,1\rangle \text{ and } \kappa(F(x),t) \le G(y) \right\}.\end{aligned}$$

Furthermore, we have

(4) $$\mathrm{Sup}\left\{ \kappa(F(x),t) \middle| t \in \langle 0,1\rangle \text{ and } \kappa(F(x),t) \le G(y) \right\} \le G(y),$$

hence by (2), (3) and (4)

(5) $$\kappa(F(x), \pi(F(x), G(y))) \le G(y),$$

hence

(6) $$\mathrm{Sup}\{\kappa(F(x), \pi(F(x), G(y))) \mid x \in U\} \le G(y),$$

hence by definition of $\Phi_{\Im}^{R}$

(7) $$\Phi_{\Im}^{R}(F)(y) \le G(y)$$

holds. ■

Theorem 2.4.11
If R = *IF* F *THEN* G *and* $\Im = [\pi, \kappa, Q]$ *is an interpretation of* R *such that*

1. $\mathrm{hgt}(F) > \mathrm{hgt}(G)$ *or* $(\mathrm{hgt}(F) \ge \mathrm{hgt}(G)$ *and* $\exists x (x \in U \wedge F(x) = \mathrm{hgt}(F)))$
2. $\forall r \left(r \in \langle 0,1\rangle \rightarrow \kappa(r,0) = 0 \wedge \kappa(r,1) = r \right)$
3. *the function* κ *is monotone and continuous in its second argument*
4. *the function* π *is defined by*

   $$\pi(r,s) =_{def} \mathrm{Sup}\left\{ t \middle| t \in \langle 0,1\rangle \wedge \kappa(r,t) \le s \right\}$$

   for every $r, s \in \langle 0,1\rangle$
5. $Q = \mathrm{Sup}$

then for every $y \in U$, $\quad G(y) \le \Phi_{\Im}^{R}(F)(y)$.

Proof
By definition of $\Phi_{\Im}^{R}$ we have to show

(1) $$G(y) \le \mathrm{Sup}\{\kappa(F(x), \pi(F(x), G(y))) \mid x \in U\},$$

hence by definition of π it is sufficient to prove

(2) $$G(y) \le \mathrm{Sup}\left\{ \kappa\left(F(x), \mathrm{Sup}\left\{ t \middle| t \in \langle 0,1\rangle \wedge \kappa(F(x),t) \le G(y) \right\}\right) \middle| x \in U \right\}.$$

Because κ is monotone in its second argument it is sufficient to show

(3) $$G(y) \leq \text{Sup}\left\{\text{Sup}\left\{\kappa(F(x),t) \middle| t \in \langle 0,1\rangle \wedge \kappa(F(x),t) \leq G(y)\right\} \middle| x \in U\right\},$$

hence it is sufficient to show

(4) $$\exists x \exists t \left(x \in U \wedge t \in \langle 0,1\rangle \wedge \kappa(F(x),t) \leq G(y) \wedge G(y) \leq \kappa(F(x),t)\right).$$

i. e.

(5) $$\exists x \exists t \left(x \in U \wedge t \in \langle 0,1\rangle \wedge \kappa(F(x),t) = G(y)\right).$$

Assumption 1 implies

(6) $$\forall y (y \in U \to \exists x (x \in U \wedge F(x) \geq G(y))).$$

Let x_0 be an element from U such that

(7) $$G(y) \leq F(x_0).$$

Then by assumption 2 we obtain

(8) $$0 = \kappa(F(x_0),0) \leq G(y) \leq \kappa(F(x_0),1) = F(x_0).$$

By assumption 3, the function κ is continuous in its second argument, hence by the intermediate value theorem there exists a $t_0 \in \langle 0,1\rangle$ such that

(9) $$F(x_0,t_0) = G(y).$$

Put $x =_{def} x_0$ and $t =_{def} t_0$, then (5) holds. ■

Remark If we weaken the assumption

$$\text{hgt}(F) > \text{hgt}(G) \quad \text{or} \quad (\text{hgt}(F) \geq \text{hgt}(G) \text{ and } \exists x (x \in U \wedge F(x) = \text{hgt}(F)))$$

to

$$\text{hgt}(F) \geq \text{hgt}(G)$$

then the proof above will not work. The reason is that the assumption

$$\text{hgt}(F) \geq \text{hgt}(G)$$

does not imply the conclusion (6).

Therefore, we have to strengthen other assumptions and to modify the proof as the following theorem shows.

Theorem 2.4.12
If $R = \text{IF}\, F\, \text{THEN}\, G$ and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. $\text{hgt}(F) \geq \text{hgt}(G)$
2. $\forall r \left(r \in \langle 0,1\rangle \to \kappa(r,0) = 0 \wedge \kappa(r,1) = r\right)$
3. *the function κ is monotone and continuous in the first as well as in the second argument*
4. *the function π is defined by*

$$\pi(r,s) =_{def} \text{Sup}\left\{t \middle| t \in \langle 0,1\rangle \wedge \kappa(r,t) \leq s\right\}$$

for every $r,s \in \langle 0,1\rangle$

5. $Q = \text{Sup}$

then for every $y \in U$*,*

$$G(y) \le \Phi^R_{\mathfrak{J}}(F)(y).$$

Proof

By assumption 1 we have

$$\text{(1)} \qquad \text{hgt}(F) \ge \text{hgt}(G),$$

hence

$$\text{(2)} \qquad \forall y (y \in U \to \text{hgt}(F) \ge G(y)).$$

Now, let y be a fixed element from U. Then by assumption 2 we obtain

$$\text{(3)} \qquad 0 = \kappa(\text{hgt}(F), 0) \le G(y) \le \kappa(\text{hgt}(F), 1) = \text{hgt}(F).$$

Assumption 3 implies the continuity of κ in its second argument, hence by the intermediate value theorem there exists a real number $t_0 \in \langle 0, 1 \rangle$ such that

$$\text{(4)} \qquad \kappa(\text{hgt}(F), t_0) = G(y).$$

Assumption 3 implies the monotonicity of κ in its first argument, hence

$$\text{(5)} \qquad \forall x (x \in U \to \kappa(F(x), t_0) \le G(y)).$$

Futhermore, assumption 3 implies the continuity of κ in its first argument, hence from (4) we get

$$\text{(6)} \qquad \text{Sup}\,\{\kappa(F(x), t_0) | x \in U\} \ge G(y).$$

Because of (5) we have

$$\text{(7)} \qquad \begin{aligned} &\text{Sup}\,\{\kappa(F(x), t_0) | x \in U\} \\ &= \text{Sup}\,\{\kappa(F(x), t_0) | x \in U \wedge \kappa(F(x), t_0) \le G(y), \} \end{aligned}$$

hence by (6)

$$\text{(8)} \qquad G(y) = \text{Sup}\,\{\kappa(F(x), t_0) | x \in U \wedge \kappa(F(x), t_0) \le G(y)\},$$

hence

$$\text{(9)} \qquad G(y) \le \text{Sup}\left\{\text{Sup}\,\{\kappa(F(x), t) | x \in U \wedge \kappa(F(x), t) \le G(y)\} \middle| t \in \langle 0, 1 \rangle\right\}.$$

Because of the "commutativity" of Sup we have

$$\text{(10)} \qquad \begin{aligned} &\text{Sup}\left\{\text{Sup}\,\{\kappa(F(x), t) | x \in U \wedge \kappa(F(x), t) \le G(y)\} \middle| t \in \langle 0, 1 \rangle\right\} \\ &= \text{Sup}\left\{\text{Sup}\left\{\kappa(F(x), t) \middle| t \in \langle 0, 1 \rangle \wedge \kappa(F(x), t) \le G(y)\right\} \middle| x \in U\right\}. \end{aligned}$$

Because of assumption 3, the function κ is monotone in its second argument, we get

$$\text{(11)} \qquad \begin{aligned} &\text{Sup}\left\{\kappa(F(x), t) \middle| t \in \langle 0, 1 \rangle \wedge \kappa(F(x), t) \le G(y)\right\} \\ &\le \kappa\left(F(x), \text{Sup}\left\{t \middle| t \in \langle 0, 1 \rangle \wedge \kappa(F(x), t) \le G(y)\right\}\right), \end{aligned}$$

hence because of monotonicity of Sup and definition of π

$$\text{(12)} \qquad \begin{aligned} &\text{Sup}\left\{\text{Sup}\left\{\kappa(F(x), t) \middle| t \in \langle 0, 1 \rangle \wedge \kappa(F(x), t) \le G(y)\right\} \middle| x \in U\right\} \\ &\le \text{Sup}\,\{\kappa(F(x), \pi(F(x), G(y))) | x \in U\}, \end{aligned}$$

From (9), (10), (12), and the definition of $\Phi^R_{\mathfrak{J}}(F)(y)$ we obtain

$$G(y) \le \Phi^R_{\mathfrak{J}}(F)(y)$$

■

Corollary 2.4.13
If R = IF F THEN G and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation of R such that

1. $\mathrm{hgt}(F) > \mathrm{hgt}(G)$ *or* ($\mathrm{hgt}(F) \geq \mathrm{hgt}(G)$ *and* $\exists x(x \in U \wedge G(x) = \mathrm{hgt}(G))$)
2. $\forall r\left(r \in \langle 0, 1\rangle \rightarrow \kappa(r, 0) = 0 \wedge \kappa(r, 1) = r\right)$
3. *the function κ is monotone and continuous in its second argument*
4. *the function π is defined by*

$$\pi(r, s) =_{def} \mathrm{Sup}\left\{t \middle| t \in \langle 0, 1\rangle \wedge \kappa(r, t) \leq s\right\}$$

   for every $r, s \in \langle 0, 1\rangle$.

5. $Q = \mathrm{Sup}$

then $\Phi^R_{\mathfrak{J}}(F) = G$, *i. e.* $\Phi^R_{\mathfrak{J}}$ *is locally correct.*

Proof By applying theorem 2.4.10 and 2.4.11. ■

Corollary 2.4.14
If R = IF F THEN G and $\mathfrak{J} = [\pi, \kappa, Q]$ is an interpretation such that

1. $\mathrm{hgt}(F) \geq \mathrm{hgt}(G)$
2. $\forall r\left(r \in \langle 0, 1\rangle \rightarrow \kappa(r, 0) = 0 \wedge \kappa(r, 1) = r\right)$
3. *the function κ is monotone and continuous as well as in its first and in its second argument*
4. *the function π is defined by*

$$\pi(r, s) =_{def} \mathrm{Sup}\left\{t \middle| t \in \langle 0, 1\rangle \wedge \kappa(r, t) \leq s\right\}$$

   for every $r, s \in \langle 0, 1\rangle$

5. $Q = \mathrm{Sup}$

then $\Phi^R_{\mathfrak{J}}(F) = G$, *i. e.* $\Phi^R_{\mathfrak{J}}$ *is locally correct.*

Proof By applying theorem 2.4.10 and 2.4.12. ■

2.5 Criteria for Global Correctness. Concepts of Continuity for Functional Operators

Now, we turn over to investigate the weakly global correctness and the global correctness of $\Phi^R_{\mathfrak{J}}$.

We state that the global correctness of $\Phi^R_{\mathfrak{J}}$ implies its weakly global correctness. Therefore in the following we shall study only the global correctness. The problem whether there exists a $\Phi^R_{\mathfrak{J}}$ such that $\Phi^R_{\mathfrak{J}}$ is weakly global correct but not global correct remains open.

Because of the above theorems it is sufficient to study the continuity of $\Phi^R_{\mathfrak{J}}$ with respect to a co-tolerance relation ρ and a set $\mathcal{F} \subseteq FP(U)$.

In order to have simple assumptions we start our considerations with the well-known CHEBYSHEV metric ρ_{C} (see example 1.1).

Obviously, the metric ρ_{C} satisfies the following

Lemma 2.5.1
For every $x \in U$*, every* $F, G \in FP(U)$ *and every real number* $c \geq 0$*,*

$$\rho_C(F,G) \leq c \text{ if and only if for every } x \in U, \quad |F(x) - G(x)| \leq c.$$

We underline that the relation between the metric ρ_C and the absolute value of real numbers expressed by this lemma will play an important role in proving theorem 2.5.3 further down.

In the proof of the following theorem we still need

Lemma 2.5.2
For every $F, G \in FP(U)$*,*

$$|\mathrm{Sup}\{F(x) | x \in U\} - \mathrm{Sup}\{G(x) | x \in U\}| \leq \mathrm{Sup}\{|F(x) - G(x)| \,|\, x \in U\}.$$

Proof Without loss of generality we can assume that

(1) $$\mathrm{Sup}\{F(x) | x \in U\} > \mathrm{Sup}\{G(y) | y \in U\}.$$

First, we show

(2) $$\forall x (x \in U \wedge F(x) \geq \mathrm{Sup}\{G(y) | y \in U\} \rightarrow F(x) - \mathrm{Sup}\{G(y) | y \in U\} \leq \mathrm{Sup}\{|F(y) - G(y)| \,|\, y \in U\}).$$

Assume

(3) $$F(x) \geq \mathrm{Sup}\{G(y) | y \in U\}.$$

Then we obtain

(4) $$\begin{aligned} & F(x) - \mathrm{Sup}\{G(y) | y \in U\} \\ & \leq F(x) - G(y) \quad \text{because of } \mathrm{Sup}\{G(y) | y \in U\} \geq G(y) \text{ for every } y \in U \\ & \leq F(x) - G(x) \quad \text{for } y =_{def} x \\ & = |F(x) - G(x)| \\ & \leq \mathrm{Sup}\{|F(x) - G(x)| \,|\, x \in U\}. \end{aligned}$$

Furthermore, we state

(5) $$\begin{aligned} & \mathrm{Sup}\{F(x) - \mathrm{Sup}\{G(y) | y \in U\} \,|\, x \in U\} \\ & = \mathrm{Sup}\{F(x) - \mathrm{Sup}\{G(y) | y \in U\} \,|\, x \in U \wedge F(x) \geq \mathrm{Sup}\{G(y) | y \in U\}\}. \end{aligned}$$

The inequation $\geq$ of (5) is trivial. In order to prove the inequation $\leq$ of (5) it is sufficient to show

(6) $$\forall x (x \in U \rightarrow \exists x' (x' \in U \wedge F(x') \geq \mathrm{Sup}\{G(y) | y \in U\} \wedge F(x') \geq F(x))).$$

Case 1 $\mathrm{Sup}\{G(y) | y \in U\} \geq F(x)$.

Because of (1)

(7) there exists an $x' \in U$ such that $F(x') \geq \mathrm{Sup}\{G(y) | y \in U\}$,

hence (6) holds.

Case 2 $F(x) \geq \mathrm{Sup}\{G(y) | y \in U\}$.

Put $x' =_{def} x$, hence (6) holds trivially.

Now, from assumption (1) we obtain

(8) $$|\mathrm{Sup}\{F(x)|x\in U\}-\mathrm{Sup}\{G(y)|y\in U\}|=\mathrm{Sup}\{F(x)|x\in U\}-\mathrm{Sup}\{G(y)|y\in U\}$$

By continuity of the subtraction $r-s$ for real numbers we get

(9) $$\mathrm{Sup}\{F(x)|x\in U\}-\mathrm{Sup}\{G(y)|y\in U\}\le \mathrm{Sup}\{F(x)-\mathrm{Sup}\{G(y)|y\in U\}\,|\,x\in U\},$$

hence by (4), (5), (8), and (9) we obtain

$$|\mathrm{Sup}\{F(x)|x\in U\}-\mathrm{Sup}\{G(y)|y\in U\}|\le \mathrm{Sup}\{|F(x)-G(x)|\,|\,x\in U\}.$$

■

Remark Using the definition of hgt and ρ_{C}, the above lemma says that for every $F,G\in FP(U)$,

$$|\mathrm{hgt}(F)-\mathrm{hgt}(G)|\le \rho_{\mathrm{C}}(F,G).$$

Theorem 2.5.3
If $R=$ IF F THEN G and $\mathfrak{J}$ is an interpretation of R such that

1. *κ is continuous*
2. *$Q=\mathrm{Sup}$*

then $\Phi^R_{\mathfrak{J}}$ is continuous with respect to ρ_{C} and $FP(U)$, i. e. $\Phi^R_{\mathfrak{J}}$ is globally correct with respect to ρ_{C} and $FP(U)$.

Proof We have to prove

(1) $$\begin{aligned}&\forall\varepsilon(\varepsilon>0\rightarrow\exists\delta(\delta>0\wedge\\ &\forall H\forall H'(H,H'\in FP(U)\wedge\rho_{\mathrm{C}}(H,H')\le\delta\rightarrow\rho_{\mathrm{C}}(\Phi^R_{\mathfrak{J}}(H),\Phi^R_{\mathfrak{J}}(H'))\le\varepsilon))).\end{aligned}$$

Assume $\varepsilon>0$. Then we want to show

(2) $$\rho_{\mathrm{C}}\left(\Phi^R_{\mathfrak{J}}(H),\Phi^R_{\mathfrak{J}}(H')\right)\le\varepsilon.$$

Because of lemma 2.5.1 it is sufficient to prove

(3) $$\left|\Phi^R_{\mathfrak{J}}(H)(y)-\Phi^R_{\mathfrak{J}}(H')(y)\right|\le\varepsilon \text{ for every } y\in U.$$

Hence by definition of $\Phi^R_{\mathfrak{J}}$ it is sufficient to show

(4) $$|\mathrm{Sup}\{\kappa(H(x),\pi(F(x),G(y)))|\,x\in U\}-\mathrm{Sup}\{\kappa(H'(x),\pi(F(x),G(y)))|\,x\in U\}|\le\varepsilon \quad \text{for every } y\in U,$$

hence by lemma 2.5.2 it is sufficient to prove

(5) $$\mathrm{Sup}\{|\kappa(H(x),\pi(F(x),G(y)))-\kappa(H'(x),\pi(F(x),G(y)))|\,|\,x\in U\}\le\varepsilon \quad \text{for every } y\in U,$$

hence by definition of Sup it is sufficient to prove

(6) $$|\kappa(H(x),\pi(F(x),G(y)))-\kappa(H'(x),\pi(F(x),G(y)))|\le\varepsilon \text{ for every } x,y\in U.$$

Because κ is continuous, we have

$$(7)\quad \forall\varepsilon(\varepsilon > 0 \to \exists\delta(\delta > 0 \wedge \forall r\forall r'\forall s\forall s'(r,r',s,s' \in \langle 0,1\rangle \wedge |r-r'| \le \delta \wedge |s-s'| \le \delta \to |\kappa(r,s)-\kappa(r',s')| \le \varepsilon))).$$

Put $r =_{def} H(x)$
$r' =_{def} H'(x)$
$s =_{def} s' =_{def} \pi(F(x),G(y))$.
Then there is a $\delta > 0$ such that the condition

$$(8)\qquad |H(x)-H'(x)| \le \delta \quad \text{for every } x \in U$$

implies (6).

Now, (1) gives the assumption

$$(9)\qquad \rho_{\mathrm{C}}(H,H') \le \delta,$$

hence by lemma 2.5.1, (9) implies (8). ■

For investigating the continuity of operators $\Phi : FP(U) \to FP(U)$ in more detail for $\mathcal{F} \subseteq FP(U)$ we define

Definition 2.5.1

1. *Φ is said to be point-wise continuous in $\mathcal{F}$*
   $=_{def} \forall\varepsilon\forall x(\varepsilon > 0 \wedge x \in U \to$
   $\exists\delta(\delta > 0 \wedge \forall F\forall G(F,G \in \mathcal{F} \wedge |F(x)-G(x)| \le \delta \to |\Phi(F)(x)-\Phi(G)(x)| \le \varepsilon)))$.
2. *Φ is said to be uniformly point-wise continuous in $\mathcal{F}$*
   $=_{def} \forall\varepsilon(\varepsilon > 0 \to \exists\delta(\delta > 0 \wedge \forall x\forall F\forall G(x \in U \wedge F,G \in \mathcal{F} \wedge |F(x)-G(x)| \le \delta$
   $\to |\Phi(F)(x)-\Phi(G)(x)| \le \varepsilon)))$.
3. *Φ is said to be* CHEBYSHEV*-continuous in $\mathcal{F}$*
   $=_{def}$ *Φ is continuous in $\mathcal{F}$ with respect to the* CHEBYSHEV *metric ρ_{C}* .

We have the following

Lemma 2.5.4

1. *Φ is uniformly point-wise continuous in $\mathcal{F}$ if and only if*
   Φ is CHEBYSHEV*-continuous in $\mathcal{F}$*
2. *The* CHEBYSHEV*-continuity of Φ in $\mathcal{F}$ implies its point-wise continuity in $\mathcal{F}$.*
3. *If U is finite then the point-wise continuity of Φ in $\mathcal{F}$ implies its* CHEBYSHEV*-continuity in $\mathcal{F}$*
4. *If U is infinite then the point-wise continuity of Φ in $\mathcal{F}$ does not imply its* CHEBYSHEV*-continuity of Φ, in general.*

Proof

ad 1 Obvious by definitions.

ad 2 Obvious by definitions.

ad 3 By assumption for every $\varepsilon > 0$ and every $x \in U$ there exists a $\delta_{\varepsilon,x}$ such that

$$\forall F\forall G\big(F,G \in \mathcal{F} \wedge |F(x)-G(x)| < \delta_{\varepsilon,x} \to |\Phi(F)(x)-\Phi(G)(x)| < \varepsilon\big).$$

$$\delta_\varepsilon =_{def} \mathrm{Sup}\left\{\delta_{\varepsilon,x} \,\middle|\, x \in U\right\}.$$

Because U is finite, δ_ε is a *finite* real number, furthermore we have

$$\delta_{\varepsilon,x} \leq \delta_\varepsilon$$

for all $x \in U$, hence δ_ε can be used to prove the CHEBYSHEV-continuity of Φ in $\mathcal{F}$.

ad 4 By analyzing the following remark, definition and theorem we get a method to construct an operator which is point-wise continuous in $\mathcal{F}(U)$ but not CHEBYSHEV-continuous in $\mathcal{F}(U)$. ■

Remark The proof of theorem 2.5.3 shows that this theorem is still valid if the function κ is only continuous in its first argument, but uniformly continuous with respect to its second argument.

For definiteness we repeat

Definition 2.5.2

1. *κ is said to be continuous in its first argument*
   $=_{def} \forall\varepsilon\forall s(\varepsilon > 0 \wedge s \in \langle 0,1\rangle \rightarrow \exists\delta(\delta > 0 \wedge \forall r\forall r'(r,r' \in \langle 0,1\rangle \wedge |r-r'| < \delta$
   $\rightarrow |\kappa(r,s)-\kappa(r',s)| < \varepsilon)))$
2. *κ is said to be uniformly continuous in its first argument with respect to its second argument*
   $=_{def} \forall\varepsilon(\varepsilon > 0 \rightarrow \exists\delta(\delta > 0 \wedge \forall r\forall r'\forall s(r,r',s \in \langle 0,1\rangle \wedge |r-r'| < \delta$
   $\rightarrow |\kappa(r,s)-\kappa(r',s)| < \varepsilon)))$.

Then, by analyzing the proof of theorem 2.5.3 we get

Theorem 2.5.5
If $R =$ IF F THEN G and $\mathfrak{I}$ is an interpretation of R such that

1. *κ is continuous in its first argument*
2. $Q = \mathrm{Sup}$

then $\Phi^R_{\mathfrak{I}}$ is point-wise continuous with respect to $FP(U)$.

Remark The continuity of operators of the form $\Phi^R_{\mathfrak{I}}$ is investigated for special $\mathfrak{I}$ and special $\mathcal{F}$ in [11].

Chapter 3

IF-THEN Rule Bases

3.1 Fundamental Concepts and Notations

We consider a fixed IF-THEN rule base

$$RB: \quad \begin{array}{c} IF\,F_1\,THEN\,G_1 \\ \vdots \\ IF\,F_n\,THEN\,G_n \end{array}$$

where $n \geq 1$ and $F_1, \dots, F_n, G_1, \dots, G_n$ are fuzzy sets on U, i. e. for every $i \in \{1, \dots, n\}$ we have $F_i : U \to \langle 0,1\rangle$ and $G_i : U \to \langle 0,1\rangle$.

For interpreting RB we fix a $(3n+4)$-tuple $\mathfrak{J}$ of the form

$$\mathfrak{J} = [\pi_1, \dots, \pi_n, \kappa_0, \kappa_1, \dots, \kappa_n, Q_0, Q_1, \dots, Q_n, \alpha, \beta]$$

where

1. $\pi_1, \dots, \pi_n, \kappa_0, \kappa_1, \dots, \kappa_n : \langle 0,1\rangle^2 \to \langle 0,1\rangle$
2. $Q_0, Q_1, \dots, Q_n : \mathfrak{P}\langle 0,1\rangle \to \langle 0,1\rangle$
3. $\alpha, \beta : \langle 0,1\rangle^n \to \langle 0,1\rangle$.

In generalization of the terminology introduced in section 2 we call $\pi_1, \dots, \pi_n$ "implication functions", $\kappa_0, \kappa_1, \dots, \kappa_n$ "combination functions", $Q_0, Q_1, \dots, Q_n$ (real) "quantifiers", and α, β "aggregation functions".

The definition of an interpretation for an IF-THEN rule base is done in three steps.

Step 1. Interpretation of the rules $IF\,F_i\,THEN\,G_i$.

For every $i \in \{1, \dots, n\}$ we define a binary fuzzy relation on U by

$$S_i(x,y) =_{def} \pi_i(F_i(x), G_i(y)) \qquad (x,y \in U).$$

We say that the implication function π_i interprets the rule $IF\,F_i\,THEN\,G_i$. The corresponding relation S_i is called "implication relation" generated by the rule $IF\,F_i\,THEN\,G_i$ and the implication function π_i.

We underline that every rule has its *own* interpretation. Of course, this general approach covers the case of a universal interpretation (i. e. $\pi_1 = \cdots = \pi_n$) as well as the case that the rule base RB contains "weighted rules", for instance.

Note that at this stage of discussion we do not assume any special properties of π_i. This will be done later for proving theorems and in considering examples.

Step 2 and step 3 depend on the principle FATI or FITA which will be used for an interpretation of the rule base being considered [7, 8, 37, 40, 46].

We define FATI $=_{def}$ **F**irst **A**ggregation **T**hen **I**nference.

Step 2 (FATI)

This step 2 consists of aggregating the defined binary fuzzy relations $S_1, \ldots, S_n$ on U to a binary fuzzy "superrelation" S_0 on U as follows

$$S_0(x,y) =_{def} \alpha(S_1(x,y), \ldots, S_n(x,y)) \qquad (x, y \in U).$$

As in step 1 we do not make any assumptions about the real function α. In several publications (see, for instance, [46]) one can find the opinion that α must be the function max or the function min. But we do not share this opinion because we think that in this way the approach is too strongly restricted.

Step 3 (FATI)

For an arbitrary "argument" F, where F is a fuzzy set on U, we infer from F and the aggregated implication relation S_0 by Q_0-κ_0-composition the fuzzy set H on U where for every $y \in U$, $H_0(y)$ is defined by $H_0(y) =_{def} Q_0(\{\kappa_0(F(x), S_0(x,y)) | x \in U\})$.

Now with respect to the interpretation $\mathfrak{I}$ we define the functional operator $FATI^{RB}_{\mathfrak{I}}$, which is a mapping $FATI^{RB}_{\mathfrak{I}} : FP(U) \to FP(U)$, as follows where $F : U \to \langle 0, 1 \rangle$ and $y \in U$:

Definition 3.1.1 (FATI)
$FATI^{RB}_{\mathfrak{I}}(F)(y) =_{def} H_0(y)$.

Consequently, we have

$$\begin{aligned}
&FATI^{RB}_{\mathfrak{I}}(F)(y)\\
&= Q_0(\{\kappa_0(F(x), S_0(x,y)) | x \in U\})\\
&= Q_0(\{\kappa_0(F(x), \alpha(\pi_1(F_1(x), G_1(y)), \ldots, \pi_n(F_n(x), G_n(y)))) | x \in U\})
\end{aligned}$$

Now, we define FITA $=_{def}$ **F**irst **I**nference **T**hen **A**ggregation.

In order to define the corresponding functional operator $FITA^{RB}_{\mathfrak{I}}$ we exchange step 2 with step 3, i. e. for a given fuzzy set $F : U \to \langle 0, 1 \rangle$ we infer from F and the "local" implication relation S_i by Q_i-κ_i-composition the fuzzy set H_i and after this we aggregate the "local" results $H_1, \ldots, H_n$ to the global result H by the aggregation function β.

That means

Step 2 (FITA) $H_i(y) =_{def} Q_i(\{\kappa_i(F(x), S_i(x,y)) | x \in U\}) \qquad (y \in U).$

Step 3 (FITA) $H(y) =_{def} \beta(H_1(y), \dots, H_n(y))$ $(y \in U)$.

Like in case of $FATI^{RB}_{\Im}$ we define

Definition 3.1.2 (FITA)
$FITA^{RB}_{\Im}(F)(y) =_{def} H(y)$.

Consequently, we have

$$
\begin{aligned}
&FITA^{RB}_{\Im}(F)(y)\\
&= \beta(H_1(y), \dots, H_n(y))\\
&= \beta(Q_1(\{\kappa_1(F(x), \pi_1(F_1(x), G_1(y)))|x \in U\}), \dots, Q_n(\{\kappa_n(F(x), \pi_n(F_n(x), G_n(y)))|x \in U\}))
\end{aligned}
$$

Example 3.1.1 In many applications so-called "crisp" inputs play an important role. We define for $F : U \to \langle 0, 1\rangle$ and $x_0 \in U$:

Definition 3.1.3
F is said to be an x_0-crisp fuzzy set on U
$=_{def}$ *$F(x_0) = 1$ and $F(x) = 0$ for every $x \in U$ with $x \neq x_0$.*

Because an x_0-crisp fuzzy set on U is uniquely determined by x_0, we denote this set by F_{x_0}.

For crisp inputs we get the following theorem.

Theorem 3.1.1
If 1. $\kappa_0, \kappa_1, \dots, \kappa_n$ are t-norms and

2. $Q_0 = Q_1 = \dots = Q_n = \mathrm{Sup}$ and

3. $\alpha = \beta$

then for every $x_0, y \in U$,

1. $FATI^{RB}_{\Im}(F_{x_0})(y) = \alpha(S_1(x_0, y), \dots, S_n(x_0, y))$
2. $FITA^{RB}_{\Im}(F_{x_0})(y) = \beta(S_1(x_0, y), \dots, S_n(x_0, y))$
3. $FATI^{RB}_{\Im}(F_{x_0})(y) = FITA^{RB}_{\Im}(F_{x_0})$.

Proof

ad 1. By definition of $FATI^{RB}_{\Im}$ we have

$$
\begin{aligned}
FATI^{RB}_{\Im}(F_{x_0}) &= Q_0\left(\left\{\kappa_0(F_{x_0}(x), \alpha(S_1(x, y), \dots, S_n(x, y)))\middle| x \in U\right\}\right)\\
&= \mathrm{Sup}(\{\kappa_0(1, \alpha(S_1(x_0, y), \dots, S_n(x_0, y)))\}\\
&\qquad \cup \{\kappa_0(0, \alpha(S_1(x, y), \dots, S_n(x, y)))|x \in U \wedge x \neq x_0\})\\
&= \alpha(S_1(x_0, y), \dots, S_n(x_0, y))
\end{aligned}
$$

because $\kappa_0(1, s) = s$, $\kappa_0(0, s) = 0$, $\mathrm{Sup}\{r, 0\} = r$.

ad 2. Analogously to assertion 1.

ad 3. Immediately from assertions 1 and 2. ■

Note that this result is independent of the interpretation of the rules $IF\, F_i\, THEN\, G_i$, i. e. the interpreting functions π_i can be chosen without any restrictions ($i \in \{1, \dots, n\}$).

By analyzing the proof of theorem 3.1.1 we obtain the following generalizations of this theorem.

Theorem 3.1.2

1. *If* $\forall s\big(s \in \langle 0,1\rangle \rightarrow \kappa_0(0,s) = 0 \wedge \kappa_0(1,s) = s\big)$
   and
   $\forall r\big(r \in \langle 0,1\rangle \rightarrow Q_0\{0,r\} = r\big)$
   then for every $x_0, y \in U$, $\quad FATI^{RB}_{\mathfrak{J}}(F_{x_0})(y) = \alpha\,(S_1(x_0,y),\dots,S_n(x_0,y))$

2. *If* *for every* $i \in \{1,\dots,n\}$,
   $\forall r\big(r \in \langle 0,1\rangle \rightarrow \kappa_i(0,s) = 0 \wedge \kappa_i(1,s) = s\big)$
   and
   $\forall r\big(r \in \langle 0,1\rangle \rightarrow Q_i\{0,r\} = r\big)$
   then for every $x_0, y \in U$, $\quad FITA^{RB}_{\mathfrak{J}}(F_{x_0})(y) = \beta\,(S_1(x_0,y),\dots,S_n(x_0,y))$

3. *If* *for every* $i \in \{0,1,\dots,n\}$,
   $\forall r\big(r \in \langle 0,1\rangle \rightarrow \kappa_i(0,s) = s \wedge \kappa_i(1,s) = s\big)$
   and
   $\forall r\big(r \in \langle 0,1\rangle \rightarrow Q_i\{0,r\} = r\big)$
   then for every $x_0 \in U$, $\quad FATI^{RB}_{\mathfrak{J}}(F_{x_0}) = FITA^{RB}_{\mathfrak{J}}(F_{x_0})$.

Remark In the past the applications of *IF−THEN* rule bases were restricted to crisp inputs of the form F_{x_0} (see definition 3.1.3), also called "singleton" inputs.

For a crisp input F_{x_0} the relations $S_1,\dots,S_n$ have the form

$$S_i(x_0,y) = \pi_i\,(F_i(x_0), G_i(y))\,.$$

Now, in some applications the vector $[F_1(x_0),\dots,F_n(x_0)]$ is called "fuzzification" of the crisp value x_0.

We underline that this term is very misleading with respect to using "fuzzy inputs" or "non-singleton" inputs (see [5, 25, 29], for instance) which newly play an increasing role in applications. But this means that a crisp value x_0 will be fuzzified by assigning a fuzzy set $F = \varphi(x_0)$ on U to x_0 where φ is defined by the block "fuzzifier" of a fuzzy controller (see page 4).

Hence, by fuzzifying in the second sense we indeed obtain a fuzzy set $F = \varphi(x_0)$. In contrast to this concept in the first sense the vector $[F_1(x_0),\dots,F_n(x_0)]$ is *no* fuzzy set in any sense, so using the term "fuzzification" in this case is very confusing.

Theorem 3.1.1 and 3.1.2 give occasion to define for $\mathcal{F} \subseteq FP(U)$:

Definition 3.1.4
$FATI^{RB}_{\mathfrak{J}}$ *and* $FITA^{RB}_{\mathfrak{J}}$ *are equivalent with respect to* $\mathcal{F}$
$=_{def}$ *For every* $F \in \mathcal{F}$, $FATI^{RB}_{\mathfrak{J}}(F) = FITA^{RB}_{\mathfrak{J}}(F)$.

Remarks to definition 3.1.4 This definition is very important both in theory and in applications. Implementations of the inference procedure $Q_i(\{\kappa_i(F(x),S(x,y))\,|\,x \in U\})$ tend to have a very high computational complexity depending on the cardinal number card U of U.

So, with respect to FATI we have the advantage that FATI requires only one inference procedure. In contrast to this, FITA requires n such procedures, i. e. its computational complexity is much higher, in general. But as we will see later, FITA has the advantage that it is much easier to check important properties, for instance, the correctness.

Example 3.1.2 Now, we discuss the "classical" MAMDANI controller. For this purpose we assume

1. $Q_0 = Q_1 = \cdots = Q_n = \text{Sup}$
2. $\kappa_0 = \kappa_1 = \cdots = \kappa_n = \min$
3. $\alpha = \beta = \max$
4. $\pi_1 = \cdots = \pi_n = \min$.

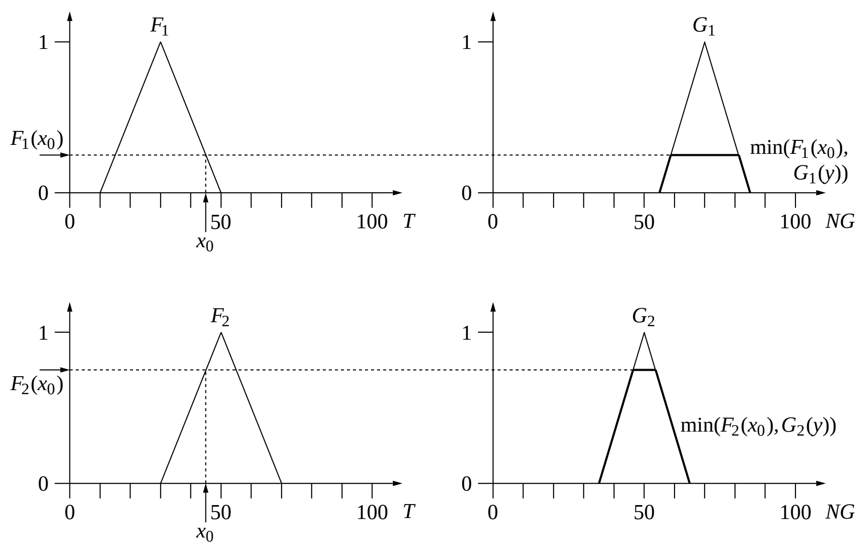
Then theorem 3.1.1 holds and for a crisp fuzzy set F_{x_0} we obtain

$$FATI_{\mathfrak{J}}^{RB}(F_{x_0})(y) = FITA_{\mathfrak{J}}^{RB}(F_{x_0})(y) = \max(\min(F_1(x_0), G_1(y)), \ldots, \min(F_n(x_0), G_n(y))).$$

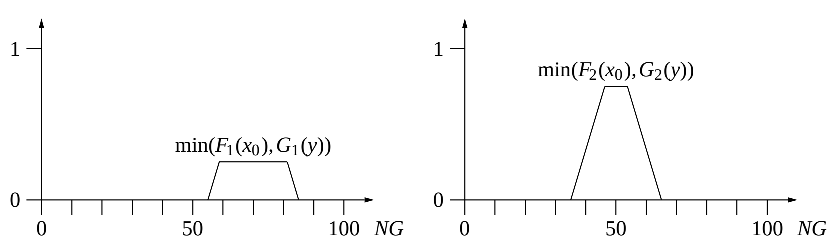
For illustration we consider the rule base

$$RB: \begin{array}{l} IF\, F_1\, THEN\, G_1 \\ IF\, F_2\, THEN\, G_2 \end{array} \qquad \text{where} \qquad \begin{array}{rl} T =_{def} & \text{Temperature} \\ NG =_{def} & \text{Natural Gas (vol/sec)} \\ x_0 =_{def} & 45 \end{array}$$

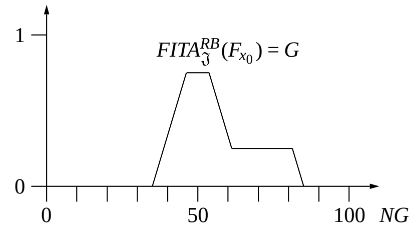
and the sets F_1, G_1, F_2 and G_2 are defined by the following figures.



Then we get



and finally $\quad FITA_{\mathfrak{J}}^{RB}(F_{x_0})(y) = G(y) = \max(\min(F_1(x_0), G_1(y)), \min(F_2(x_0), G_2(y)))$



Example 3.1.3 The LARSEN controller

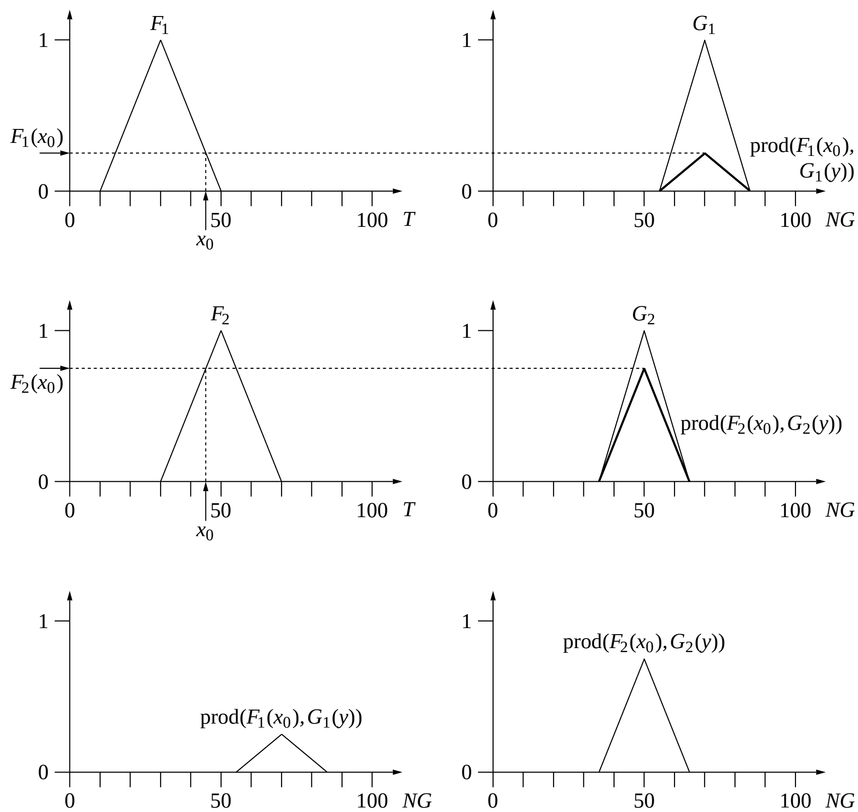
Assumption

1. $Q_0 = Q_1 = \cdots = Q_n = \text{Sup}$
2. $\kappa_0 = \kappa_1 = \cdots = \kappa_n = \min$
3. $\alpha = \beta = \max$
4. $\pi_1 = \cdots = \pi_n = \text{prod}$ where $\text{prod}(r,s) = r \cdot s$.

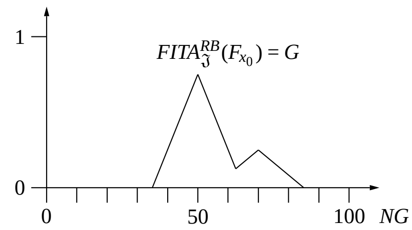
Then theorem 3.1.1 holds and for a crisp fuzzy set F_{x_0} we obtain

$$FATI^{RB}_{\mathfrak{I}}(F_{x_0})(y) = FITA^{RB}_{\mathfrak{I}}(F_{x_0})(y) = \max(\text{prod}(F_1(x_0), G_1(y)), \ldots, \text{prod}(F_n(x_0), G_n(y))).$$

For illustration we choose the same rule base RB as in example 2. Then we get



and finally $\quad FITA^{RB}_{\mathfrak{J}}(F_{x_0})(y) = G(y) = \max(\mathrm{prod}(F_1(x_0), G_1(y)), \mathrm{prod}(F_2(x_0), G_2(y)))$



3.2 On some Concepts of Correctness for Rule Bases

3.2.1 The Rule-wise Correctness of a Rule Base

We recall that for the given IF-THEN rule base

$$RB: \quad \begin{array}{c} IF\, F_1\, THEN\, G_1 \\ \vdots \\ IF\, F_n\, THEN\, G_n \end{array}$$

and the given interpretation

$$\mathfrak{J} = [\pi_1, \ldots, \pi_n, \kappa_0, \kappa_1, \ldots, \kappa_n, Q_0, Q_1, \ldots, Q_n, \alpha, \beta]$$

of RB we have defined $S_i(x,y) =_{def} \pi_i(F_i(x), G_i(y))$ for $x,y \in U$. Now, for arbitrary $F : U \to \langle 0,1 \rangle$ and $y \in U$ we put

Definition 3.2.1
$(F \odot S_i)(y) =_{def} Q_i(\{\kappa_i(F(x), S_i(x,y)) | x \in U\})$

Using this "product" $\odot$ we define

Definition 3.2.2
RB is said to be ***rule-wise correct*** *with respect to* $\mathfrak{J} =_{def}$ *For every* $i \in \{1, \ldots, n\}$, $F_i \odot S_i = G_i$.

Obviously, definition 3.2.2 is a generalization to rule bases of the correctness of one rule (see definition 2.4.1).

The following five theorems state the rule-wise correctness of IF-THEN rule bases with respect to $\mathfrak{J}$ under sufficient assumptions.

Theorem 3.2.1
If for every $i \in \{1, \ldots, n\}$,

1. π_i *and* κ_i *are t-norms*
2. $Q_i = \mathrm{Sup}$
3. *at least one of the following conditions is fulfilled*
   - *3.1.* $\mathrm{hgt}(F_i) \geq \mathrm{hgt}(G_i)$ *and* $\pi_i = \kappa_i = \min$ *or*
   - *3.2.* $\mathrm{hgt}(F_i) = 1$ *and* π_i *and* κ_i *are continuous* *or*
   - *3.3. there is an* $x \in U$ *with* $F_i(x) = 1$

then RB is rule-wise correct with respect to $\mathfrak{I}$.

Proof Application of corollary 2.4.3. ■

Theorem 3.2.2
If for every $i \in \{1, \dots, n\}$,

1. $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \pi_i(r,s) \le \min(r,s) \wedge \kappa_i(r,s) \le \min(r,s)\big)$
2. $Q_i = \mathrm{Sup}$
3. *at least one of the following three conditions is satisfied*
   - 3.1. $\mathrm{hgt}(F_i) \ge \mathrm{hgt}(G_i)$ *and* $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \min(r,s) \le \pi_i(r,s) \wedge \min(r,s) \le \kappa_i(r,s)\big)$
   - 3.2. $\mathrm{hgt}(F_i) = 1$*, π_i is continuous in its first argument, κ_i is continuous, and* $\forall s \big(s \in \langle 0,1\rangle \to \pi_i(1,s) = \kappa_i(1,s) = s\big)$
   - 3.3. $\exists x (x \in U \wedge F_i(x) = 1) \wedge \forall s \big(s \in \langle 0,1\rangle \to \pi_i(1,s) = \kappa_i(1,s) = s\big)$

then RB is rule-wise correct with respect to $\mathfrak{I}$.

Proof Application of corollary 2.4.8. ■

Theorem 3.2.3
If for every $i \in \{1, \dots, n\}$,

1. $\forall s \big(s \in \langle 0,1\rangle \to \pi_i(1,s) = \kappa_i(1,s) = s\big)$
2. *the functins π_i and κ_i are monotone in their first argument*
3. *at least one of the following three conditions is satisfied*
   - 3.1. $\mathrm{hgt}(F_i) \ge \mathrm{hgt}(G_i)$ *and* $\forall r \forall s \big(r, s \in \langle 0,1\rangle \to \min(r,s) \le \pi_i(r,s) \wedge \min(r,s) \le \kappa_i(r,s)\big)$
   - 3.2. $\mathrm{hgt}(F_i) = 1$*, π_i is continuous in its first argument, and κ_i is continuous*
   - 3.3. $\exists x (x \in U \wedge F_i(x) = 1)$

then RB is rule-wise correct with respect to $\mathfrak{I}$.

Proof By applying corollary 2.4.9. ■

Theorem 3.2.4
If for every $i \in \{1, \dots, n\}$,

1. $\mathrm{hgt}(F_i) > \mathrm{hgt}(G_i)$ *or* $(\mathrm{hgt}(F_i) \ge \mathrm{hgt}(G_i)$ *and* $\exists x (x \in U \wedge G_i(x) = \mathrm{hgt}(G_i)))$
2. $\forall r \big(r \in \langle 0,1\rangle \to \kappa_i(r,0) = 0 \wedge \kappa_i(r,1) = r\big)$
3. *the function κ_i is monotone and continuous in its second argument*
4. *π_i is defined by* $\pi(r,s) =_{def} \mathrm{Sup}\big\{t \,\big|\, t \in \langle 0,1\rangle \wedge \kappa(r,t) \le s\big\}$ *for every $r, s \in \langle 0,1\rangle$*
5. $Q_i = \mathrm{Sup}$

then RB is rule-wise correct with respect to $\mathfrak{I}$.

Proof Application of corollary 2.4.13. ■

Theorem 3.2.5
If for every $i \in \{1, \dots, n\}$,

1. $\text{hgt}(F_i) \geq \text{hgt}(G_i)$

2. $\forall r \left(r \in \langle 0,1 \rangle \rightarrow \kappa(r,0) = 0 \wedge \kappa(r,1) = r \right)$

3. the function κ_i is monotone and continuous in its first as well as in its second argument

4. the function π_i is defined by

$$\pi_i(r,s) =_{def} \text{Sup} \left\{ t \middle| t \in \langle 0,1 \rangle \wedge \kappa_i(r,t) \leq s \right\}$$

for every $r,s \in \langle 0,1 \rangle$.

5. $Q_i = \text{Sup}$

then RB is rule-wise correct with respect to $\mathfrak{I}$.

Proof By applying corollary 2.4.14. ■

3.2.2 The Local Correctness of FATI and FITA

Independently of the question whether $FATI_{\mathfrak{I}}^{RB}$ and $FITA_{\mathfrak{I}}^{RB}$ are equivalent with respect to a set $\mathcal{F} \subseteq FP(U)$ we define for $\Phi \in \left\{ FATI_{\mathfrak{I}}^{RB}, FITA_{\mathfrak{I}}^{RB} \right\}$:

Definition 3.2.3
Φ is said to be ***locally correct*** $=_{def}$ *For every $i \in \{1, \ldots, n\}$, $\Phi(F_i) = G_i$.*

3.2.3 Versions of the Global Correctness of FATI and FITA

Let ρ be a co-tolerance relation on $FP(U)$. We assume that we have an $\mathcal{F} \subseteq FP(U)$ with $F_1, \ldots, F_n \in \mathcal{F}$. As in section 3.2.2 *independently* of the equivalence of $FATI_{\mathfrak{I}}^{RB}$ and $FITA_{\mathfrak{I}}^{RB}$ (in $\mathcal{F}$) for $\Phi \in \left\{ FATI_{\mathfrak{I}}^{RB}, FITA_{\mathfrak{I}}^{RB} \right\}$ we define:

Definition 3.2.4

1. Φ is said to be ***weakly globally correct*** *with respect to ρ in $\mathcal{F}$*
$=_{def}$ *1.1. Φ is locally correct*

1.2. Φ is continuous in $F_1, \ldots, F_n$ with respect to ρ and $\mathcal{F}$.

2. Φ is said to be ***globally correct*** *with respect to ρ in $\mathcal{F}$*
$=_{def}$ *2.1. Φ is locally correct*

2.2. Φ is continuous in all "points" of $\mathcal{F}$ with respect to ρ and $\mathcal{F}$.

3.3 Criteria for Local Correctness of FITA and FATI

3.3.1 On the Local Correctness of FITA

Referring to our remarks to definition 3.1.4 we start our considerations with the principle FITA because the investigation of $FITA_{\mathfrak{I}}^{RB}$ with respect to local correctness is easier and more successful than the studying of the same question for $FATI_{\mathfrak{I}}^{RB}$.

Theorem 3.3.1
If 1. RB is rule-wise correct with respect to $\mathfrak{I}$

2. $\beta = \max$

3. *for every* $i, j \in \{1, \dots, n\}$, $F_i \textcircled{j} S_j \subseteq F_i \textcircled{i} S_i$

then $FITA_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof By definition of $FITA_{\mathfrak{J}}^{RB}$ we have for $i \in \{1, \dots, n\}$ and $y \in U$:

$$FITA_{\mathfrak{J}}^{RB}(F_i)(y) = \beta((F_i \textcircled{1} S_1)(y), \dots, (F_i \textcircled{n} S_n)(y)).$$

By assumption 3 we get for every $i, j \in \{1, \dots, n\}$ where $y \in U$

$$(F_i \textcircled{j} S_j)(y) \leq (F_i \textcircled{i} S_i)(y),$$

hence by assumptions 2 and 1

$$FITA_{\mathfrak{J}}^{RB}(F_i)(y) = (F_i \textcircled{i} S_i)(y) = G_i(y).$$

■

Remark The assumptions 2 and 3 are sufficient but not necessary. Instead of these assumptions even the following assumption is sufficient:

$$\beta((F_i \textcircled{1} S_1)(y), \dots, (F_i \textcircled{n} S_n)(y)) = (F_i \textcircled{i} S_i)(y) \qquad (i \in \{1, \dots, n\}, y \in U).$$

“Dual” to theorem 3.3.1 the following theorem holds:

Theorem 3.3.2
If 1. *RB is rule-wise correct with respect to* $\mathfrak{J}$

2. $\beta = \min$

3. *for every* $i, j \in \{1, \dots, n\}$, $F_i \textcircled{i} S_i \subseteq F_i \textcircled{j} S_j$

then $FITA_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof As for theorem 3.3.1. ■

Remark In theorem 3.3.2 the assumptions 2 and 3 are again sufficient but not necessary. Instead of these assumptions even the same assumption

$$\beta((F_i \textcircled{1} S_1)(y), \dots, (F_i \textcircled{n} S_n)(y)) = (F_i \textcircled{i} S_i)(y) \qquad (i \in \{1, \dots, n\}, y \in U)$$

as for theorem 3.3.1 is sufficient.

Theorem 3.3.3
If 1. *the assumptions of one of the theorems 3.2.1, 3.2.2, 3.2.3, 3.2.4, and 3.2.5 are satisfied*

2. *for every* $i, j \in \{1, \dots, n\}$, $F_i \textcircled{j} S_j \subseteq F_i \textcircled{i} S_i$

3. $\beta = \max$

then $FITA_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof By applying theorem 3.3.1 and the theorems quoted above under 1. ■

Theorem 3.3.4
If 1. *the assumptions of one of the theorems 3.2.1, 3.2.2, 3.2.3, 3.2.4, and 3.2.5 are satisfied*

2. *for every* $i, j \in \{1, \dots, n\}$, $F_i \textcircled{i} S_i \subseteq F_i \textcircled{j} S_j$

3. $\beta = \min$

then $FITA_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof By applying theorem 3.3.2 and the theorems quoted above under 1. ■

Remark Analogous to theorems 3.3.1 and 3.3.2 we can state that in theorems 3.3.3 and 3.3.4 the assumptions 4 and 5 are sufficient but not necessary. In both theorems we can replace these assumptions by the assumption

$$\beta((F_i \textcircled{1} S_1)(y), \ldots, (F_i \textcircled{n} S_n)(y)) = (F_i \textcircled{i} S_i)(y) \qquad (\in \{1, \ldots, n\}, y \in U)$$

which is also sufficient.

3.3.2 On the Local Correctness of FATI

Once more referring to the remarks to definition 3.1.4 we shall see that the investigation whether $FATI_{\mathfrak{J}}^{RB}$ is locally correct or not is much harder than for $FITA_{\mathfrak{J}}^{RB}$.

Theorem 3.3.5
If 1. RB is rule-wise correct with respect to $\mathfrak{J}$

2. $\alpha = \max$
3. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ *and* κ_0 *is a t-norm*
4. $Q_0 = \mathrm{Sup}$
5. *for every* $i, j \in \{1, \ldots, n\}$, $F_i \textcircled{0} S_j \subseteq F_i \textcircled{0} S_i$

then $FATI_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof We get

$$FATI_{\mathfrak{J}}^{RB}(F)(y) = Q_0(\{\kappa_0(F(x), \alpha(S_1(x,y), \ldots, S_n(x,y))) \mid x \in U\})$$

by definition, hence by assumptions 2 and 4

$$= \mathrm{Sup}\{\kappa_0(F(x), \max(S_1(x,y), \ldots, S_n(x,y))) \mid x \in U\}.$$

Because κ_0 is a t-norm we have for every $r, s_1, \ldots, s_n \in \langle 0, 1\rangle$,

$$\kappa_0(r, \max(s_1, \ldots, s_n)) = \max(\kappa_0(r, s_1), \ldots, \kappa_0(r, s_n)).$$

Consequently we obtain

$$FATI_{\mathfrak{J}}^{RB}(F)(y) = \mathrm{Sup}\{\max(\kappa_0(F(x), S_1(x,y)), \ldots, \kappa_0(F(x), S_n(x,y))) \mid x \in U\}.$$

Now, because we can exchange Sup for max we get

$$FATI_{\mathfrak{J}}^{RB}(F)(y) = \max(\mathrm{Sup}\{\kappa_0(F(x), S_1(x,y)) \mid x \in U\}, \ldots, \mathrm{Sup}\{\kappa_0(F(x), S_n(x,y)) \mid x \in U\})$$

and

$$= \max((F \textcircled{0} S_1)(y), \ldots, (F \textcircled{0} S_n)(y))$$

by definition, hence for $F = F_i$

$$FATI_{\mathfrak{J}}^{RB}(F_i)(y) = \max((F_i \textcircled{0} S_1)(y), \ldots, (F_i \textcircled{0} S_n)(y))$$

thus by assumptions 5, 3 and finally 1

$$= (F_i \textcircled{0} S_i)(y) = (F_i \textcircled{i} S_i)(y) = G_i(y).$$

■

Remarks

1. Theorem 3.3.5 also holds if we replace assumption

   3. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ and κ_0 is a t-norm

   by

   3′. κ_0 is a t-norm

   and replace assumption

   5. for every $i, j \in \{1, \ldots, n\}, F_i \circledcirc S_j \subseteq F_i \circledcirc S_i$

   by

   5′. for every $i, j \in \{1, \ldots, n\}, F_i \circledcirc S_j \subseteq F_i \odot S_i$.

2. We can state that the assumptions 2 and 5 in theorem 3.3.5 are sufficient but not necessary. We can even replace these assumptions by

$$\alpha((F_i \circledcirc S_1)(y), \ldots, (F_i \circledcirc S_n)(y)) = (F_i \circledcirc S_i)(y)$$

   for every $i \in \{1, \ldots, n\}$ and $y \in U$ which is also sufficient.

3. If we have replaced assumption 3 by 3′ and assumption 5 by 5′ then we can together replace 3′ and 5′ by

$$\alpha((F_i \circledcirc S_1)(y), \ldots, (F_i \circledcirc S_n)(y)) = (F_i \odot S_i)(y). \qquad (i \in \{1, \ldots, n\}, y \in U).$$

Now, we shall discuss the case that the aggregation function α is the minimum function.

Before we shall formulate and prove the theorem in question we prove the following theorem on the comparability of $FATI^{RB}_{\mathfrak{J}}$ with $FITA^{RB}_{\mathfrak{J}}$.

Theorem 3.3.6
If 1. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ *and* κ_0 *is a t-norm*

2. $Q_0 = Q_1 = \cdots = Q_n = \mathrm{Sup}$

3. $\alpha = \beta = \min$

then $FATI^{RB}_{\mathfrak{J}}(F) \subseteq FITA^{RB}_{\mathfrak{J}}(F)$ *for every* $F : U \to \langle 0, 1\rangle$.

Proof Because κ_0 is a t-norm we have for every $r, s_1, \ldots, s_n \in \langle 0, 1\rangle$,

$$\kappa_0(r, \min(s_1, \ldots, s_n)) = \min(\kappa_0(r, s_1), \ldots, \kappa_0(r, s_n)).$$

Then we get by definition of $FATI^{RB}_{\mathfrak{J}}$ and by assumptions 1 and 3

$$\begin{aligned} FATI^{RB}_{\mathfrak{J}}(F)(y) &= \mathrm{Sup}\{\kappa_0(F(x), \min(S_1(x,y), \ldots, S_n(x,y))) | x \in U\} \quad , \quad \text{thus} \\ &= \mathrm{Sup}\{\min(\kappa_0(F(x), S_1(x,y)), \ldots, \kappa_0(F(x), S_n(x,y))) | x \in U\} \end{aligned}$$

because κ_0 is a t-norm, so

$$\leq \min(\mathrm{Sup}\{\kappa_0(F(x), S_1(x,y)) | x \in U\}, \ldots, \mathrm{Sup}\{\kappa_0(F(x), S_n(x,y)) | x \in U\})$$

because of

$$\mathrm{Sup}\{\min(F(x), G(x)) | x \in U\} \leq \min(\mathrm{Sup}\{F(x) | x \in U\}, \mathrm{Sup}\{G(x) | x \in U\})$$

where F and G are arbitrary fuzzy sets on U, hence

$$FATI^{RB}_{\mathfrak{J}}(F)(y) \leq \min((F \textcircled{0} S_1)(y), \ldots , (F \textcircled{0} S_n)(y)).$$

Because of assumptions 1, 2 and 3 we have $F \textcircled{0} S_i = F \textcircled{i} S_i$ for every $i \in \{1, \ldots , n\}$, thus for every $y \in U$

$$\min((F \textcircled{0} S_1)(y), \ldots , (F \textcircled{0} S_n)(y)) = \min((F \textcircled{1} S_1)(y), \ldots , (F \textcircled{n} S_n)(y)) = FITA^{RB}_{\mathfrak{J}}(F)(y),$$

hence $FATI^{RB}_{\mathfrak{J}}(F) \subseteq FITA^{RB}_{\mathfrak{J}}(F)$. ■

Theorem 3.3.7
If 1. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ *and* κ_0 *is a t-norm*

2. $Q_0 = Q_1 = \cdots = Q_n = \mathrm{Sup}$

3. $\alpha = \beta = \min$

4. *RB is rule-wise correct with respect to* $\mathfrak{J}$

5. *for every* $i, j \in \{1, \ldots , n\}$, $F_i \textcircled{i} S_i \subseteq F_i \textcircled{j} S_j$

then $FATI^{RB}_{\mathfrak{J}}(F_i) \subseteq G_i$.

Proof From assumptions 1, 2 and 3 we get by theorem 3.3.6

$$FATI^{RB}_{\mathfrak{J}}(F_i) \subseteq FITA^{RB}_{\mathfrak{J}}(F_i) \text{ for every } i \in \{1, \ldots , n\}.$$

From assumptions 3, 4 and 5 we get for every $i \in \{1, \ldots , n\}$,

$$FITA^{RB}_{\mathfrak{J}}(F_i) = G_i,$$

hence theorem 3.3.7 holds. ■

If we can replace the conclusion $FATI^{RB}_{\mathfrak{J}}(F_i) \subseteq FITA^{RB}_{\mathfrak{J}}(F_i)$ in theorem 3.3.6 by $FATI^{RB}_{\mathfrak{J}}(F_i) = FITA^{RB}_{\mathfrak{J}}(F_i)$ for every $i \in \{1, \ldots , n\}$ then we get a further correctness theorem for $FATI^{RB}_{\mathfrak{J}}$. Now, let us prove $FATI^{RB}_{\mathfrak{J}}(F_i) = FITA^{RB}_{\mathfrak{J}}(F_i)$ for every $i \in \{1, \ldots , n\}$.

Theorem 3.3.8
If 1. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ *and* κ_0 *is a t-norm*

2. $Q_0 = Q_1 = \cdots = Q_n = \mathrm{Sup}$

3. $\alpha = \beta = \min$

4. *for every* $i \in \{1, \ldots , n\}$ *and every* $y \in U$ *there is a* $j \in \{1, \ldots , n\}$ *such that for every* $x \in U$,

$$\kappa_0(F_i(x), S_j(x,y)) \leq \mathrm{Sup}\{\min(\kappa_0(F_i(x), S_1(x,y)), \ldots , \kappa_0(F_i(x), S_n(x,y))) \mid x \in U\}$$

then $FATI^{RB}_{\mathfrak{J}}(F_i) = FITA^{RB}_{\mathfrak{J}}(F_i)$ *for every* $i \in \{1, \ldots , n\}$.

Proof We start the proof as the one for theorem 3.3.6.

Then we have for every $i \in \{1, \ldots , n\}$,

$$FATI^{RB}_{\mathfrak{J}}(F_i)(y) = \mathrm{Sup}\{\min(\kappa_0(F_i(x), S_1(x,y)), \ldots , \kappa_0(F_i(x), S_n(x,y))) \mid x \in U\}.$$

Now, assumption 4 implies for certain $j \in \{1, \ldots, n\}$ that

$$\begin{aligned}\operatorname{Sup}\Big\{\kappa_0(F_i(x), S_j(x,y))\Big| x \in U\Big\} \\ \leq \operatorname{Sup}\{\min(\kappa_0(F_i(x), S_1(x,y)), \ldots, \kappa_0(F_i(x), S_n(x,y)))| x \in U\},\end{aligned}$$

hence

$$\begin{aligned}\min(\operatorname{Sup}\{\kappa_0(F_i(x), S_1(x,y))| x \in U\}, \ldots, \operatorname{Sup}\{\kappa_0(F_i(x), S_n(x,y))| x \in U\}) \\ \leq \operatorname{Sup}\{\min(\kappa_0(F_i(x), S_1(x,y)), \ldots, \kappa_0(F_i(x), S_n(x,y)))| x \in U\}.\end{aligned}$$

The rest of the proof follows the proof of theorem 3.3.6, hence we get

$$FATI_{\mathfrak{J}}^{RB}(F_i) = FITA_{\mathfrak{J}}^{RB}(F_i) \text{ for every } i \in \{1, \ldots, n\}.$$

■

Theorem 3.3.9

If 1. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$ *and* κ_0 *is a t-norm*

2. $Q_0 = Q_1 = \cdots = Q_n = \operatorname{Sup}$

3. $\alpha = \beta = \min$

4. *RB is rule-wise correct with respect to* $\mathfrak{J}$

5. *for every* $i, j \in \{1, \ldots, n\}$ $F_i \textcircled{i} S_i \subseteq F_i \textcircled{j} S_j$

6. *for every* $i \in \{1, \ldots, n\}$ *and every* $y \in U$ *there is a* $j \in \{1, \ldots, n\}$ *such that for every* $x \in U$,

$$\kappa_0(F_i(x), S_j(x,y)) \leq \operatorname{Sup}\{\min(\kappa_0(F_i(x), S_1(x,y)), \ldots, \kappa_0(F_i(x), S_n(x,y)))| x \in U\}$$

then for every $i \in \{1, \ldots, n\}$, $FATI_{\mathfrak{J}}^{RB}(F_i) = G_i$, *i. e.* $FATI_{\mathfrak{J}}^{RB}$ *is locally correct.*

Proof Application of theorems 3.3.7 and 3.3.8. ■

3.4 On the Continuity of FATI and FITA

3.4.1 Discussing the Operator FATI

Remember that $FATI_{\mathfrak{J}}^{RB}$ is defined by

$$FATI_{\mathfrak{J}}^{RB}(H)(y) =_{def} Q_0(\{\kappa_0(H(x), S_0(x,y))| x \in U\})$$

where

$$S_0(x,y) =_{def} \alpha(\pi_1(F_1(x), G_1(y)), \ldots, \pi_n(F_n(x), G_n(y))).$$

Because the "superrelation" S_0 does not depend on F the operator $FATI_{\mathfrak{J}}^{RB}$ has the same form as the operator $\Phi_{\mathfrak{J}}^{R}$ defined by a single rule, i. e.

$$\Phi_{\mathfrak{J}}^{R}(h)(y) = Q(\{\kappa(H(x), \pi(F(x), G(y)))| x \in U\}).$$

Therefore we can adopt the corresponding results from chapter 2, in particular the theorems 2.5.3 and 2.5.5.

Theorem 3.4.1
If RB is a rule base and $\mathfrak{I}$ is an interpretation of RB such that

1. κ_0 *is continuous*
2. $Q_0 = \mathrm{Sup}$

then $FATI_{\mathfrak{I}}^{RB}$ *is* CHEBYSHEV-*continuous with respect to* $FP(U)$.

As in theorem 2.5.3 we can generalize the assumption "κ is continuous" to "κ is uniformly continuous in its first argument with respect to its second argument".

Furthermore, we obtain

Theorem 3.4.2
If RB is a rule base and $\mathfrak{I}$ is an interpretation of RB such that

1. κ_0 *is continuous in its first argument*
2. $Q_0 = \mathrm{Sup}$

then $FATI_{\mathfrak{I}}^{RB}$ *is point-wise continuous with respect to* $FP(U)$.

3.4.2 Discussing the Operator FITA

Remember that $FITA_{\mathfrak{I}}^{RB}$ is defined by

$$FITA_{\mathfrak{I}}^{RB}(H)(y) =_{def} \beta(H_1(y), \dots, H_n(y))$$

where for every $i \in \{1, \dots, n\}$

$$H_i(y) =_{def} Q_i(\{\kappa_i(H(x), \pi_i(F_i(x), G_i(y))) \mid x \in U\}).$$

Obviously, each H_i is defined by a single rule $R_i = IF\, F_i\, THEN\, G_i$, hence we can apply theorem 2.5.3 and 2.5.5 to each operator $\Phi_{\mathfrak{I}}^{R_i}$. Assuming additionally the continuity of β, we get the following theorems.

Theorem 3.4.3
If RB is a rule base and $\mathfrak{I}$ is an interpretation of RB such that

1. *for every* $i \in \{1, \dots, n\}$, κ_i *is continuous*
2. *for every* $i \in \{1, \dots, n\}$, $Q_i = \mathrm{Sup}$
3. β *is continuous*

then $FITA_{\mathfrak{I}}^{RB}$ *is* CHEBYSHEV-*continuous with respect to* $FP(U)$.

As in theorem 2.5.3 and 3.4.1 we can weaken assumption 1 to "κ is uniformly continuous in the first argument with respect to its second".

Theorem 3.4.4
If RB is a rule base and $\mathfrak{I}$ is an interpretation of RB such that

1. *for each* $i \in \{1, \dots, n\}$, κ_i *is continuous in the first argument*
2. *for each* $i \in \{1, \dots, n\}$, $Q_i = \mathrm{Sup}$
3. β *is continuous*

then $FITA_{\mathfrak{I}}^{RB}$ *is point-wise continuous with respect to* $FP(U)$.

3.5 On the Equivalence of FATI and FITA

We generalize theorem 3.3.6 as follows:

Theorem 3.5.1
If 1. $Q_0 = Q_1 = \cdots = Q_n = \mathrm{Sup}$

2. for every $r, s_1, \ldots, s_n \in \langle 0,1\rangle$, $\kappa_0(r, \alpha(s_1,\ldots,s_n)) \le \beta(\kappa_1(r,s_1),\ldots,\kappa_n(r,s_n))$

3. β *is monotone*

then for every $F : U \to \langle 0,1\rangle$, $FATI^{RB}_{\mathfrak{J}}(F) \subseteq FITA^{RB}_{\mathfrak{J}}(F)$.

Proof We obtain

$$FATI^{RB}_{\mathfrak{J}}(F)(y) = Q_0(\{\kappa_0(F(x), \alpha(S_1(x,y),\ldots,S_n(x,y)))|x \in U\})$$

by definition of $FATI^{RB}_{\mathfrak{J}}$, hence

$$\le \mathrm{Sup}\left\{\beta(\kappa_1(F(x),S_1(x,y)),\ldots,\kappa_n(F(x),S_n(x,y)))\middle| x \in U\right\}$$

by assumptions 1, 2 and the monotonicity of Sup. Furthermore, because β is monotone by assumption 3 and Sup fulfills

$$F(x) \le \mathrm{Sup}\{F(x)|x \in U\} \text{ and } \mathrm{Sup}\{F(x)|x \in U\} \le c \text{ if } F(x) \le c \text{ for every } x \in U$$

for an arbitrary $F : U \to \langle 0,1\rangle$, we get

$$\begin{aligned}
&\mathrm{Sup}\left\{\beta(\kappa_1(F(x),S_1(x,y)),\ldots,\kappa_n(F(x),S_n(x,y)))\middle| x \in U\right\}\\
&\le \beta(\mathrm{Sup}\{\kappa_1(F(x),S_1(x,y))|x\in U\},\ldots,\mathrm{Sup}\{\kappa_n(F(x),S_n(x,y))|x \in U\})\\
&= FITA^{RB}_{\mathfrak{J}}(F)(y) \qquad \text{because of assumption 1,}
\end{aligned}$$

hence, we get $FATI^{RB}_{\mathfrak{J}}(F) \subseteq FITA^{RB}_{\mathfrak{J}}(F)$. ■

Remark Theorem 3.5.1 can be interpreted in the form that "$FATI^{RB}_{\mathfrak{J}}$ is more specific than $FITA^{RB}_{\mathfrak{J}}$ or is equivalent to $FITA^{RB}_{\mathfrak{J}}$." The following theorem will express that "$FITA^{RB}_{\mathfrak{J}}$ is more specific than $FATI^{RB}_{\mathfrak{J}}$ or equivalent to $FATI^{RB}_{\mathfrak{J}}$."

Theorem 3.5.2
If 1. $Q_0 = Q_1 = \cdots = Q_n = \mathrm{Sup}$

2′. for every $r, s_1, \ldots, s_n \in \langle 0,1\rangle$, $\beta(\kappa_1(r,s_1),\ldots,\kappa_n(r,s_n)) \le \kappa_0(r, \alpha(s_1,\ldots,s_n))$

3′. for every $H_1,\ldots,H_n : U \to \langle 0,1\rangle$,

$$\begin{aligned}
&\beta(\mathrm{Sup}\{H_1(x)|x\in U\},\ldots,\mathrm{Sup}\{H_n(x)|x\in U\})\\
&\qquad\qquad \le \mathrm{Sup}\left\{\beta(H_1(x),\ldots,H_n(x))\middle| x\in U\right\}
\end{aligned}$$

then for every $F : U \to \langle 0,1\rangle$, $FITA^{RB}_{\mathfrak{J}}(F) \subseteq FATI^{RB}_{\mathfrak{J}}(F)$.

Proof We obtain

$$FITA^{RB}_{\mathfrak{J}}(F)(y) = \beta(Q_1\{\kappa_1(F(x),S_1(x,y))|x \in U\},\ldots,Q_n\{\kappa_n(F(x),S_n(x,y))|x\in U\})$$

by definition of $FITA^{RB}_{\mathfrak{J}}$, hence

$$= \beta(\mathrm{Sup}\{\kappa_1(F(x),S_1(x,y))|x \in U\},\dots,\mathrm{Sup}\{\kappa_n(F(x),S_n(x,y))|x \in U\})$$

by assumption 1, thus

$$\leq \mathrm{Sup}\left\{\beta(\kappa_1(F(x),S_1(x,y)),\dots,\kappa_n(F(x),S_n(x,y)))\middle|x \in U\right\}$$

by assumption 3′, so

$$\leq \mathrm{Sup}\{\kappa_0(F(x),\alpha(S_1(x,y),\dots,S_n(x,y)))|x \in U\}$$

by assumption 2′ and the monotonicity of Sup

$$= FATI^{RB}_{\mathfrak{J}}(F)(y)$$

by definition of $FATI^{RB}_{\mathfrak{J}}$. Consequently, we get $FATI^{RB}_{\mathfrak{J}}(F)(y) \leq FITA^{RB}_{\mathfrak{J}}(F)(y)$ for every $y \in U$, i. e. $FATI^{RB}_{\mathfrak{J}}(F) \subseteq FITA^{RB}_{\mathfrak{J}}(F)$. ■

Now, we formulate the following theorem on the equivalence of $FATI^{RB}_{\mathfrak{J}}$ and $FITA^{RB}_{\mathfrak{J}}$.

Theorem 3.5.3
If the assumptions 1, 2, 3 and 2′, 3′ of theorems 3.5.1 and 3.5.2 are fulfilled then for every $F : U \to \langle 0,1\rangle$, $FATI^{RB}_{\mathfrak{J}}(F) = FITA^{RB}_{\mathfrak{J}}(F)$, i. e. $FATI^{RB}_{\mathfrak{J}}$ and $FITA^{RB}_{\mathfrak{J}}$ are equivalent with respect to the set of all fuzzy sets on U (see definition 3.1.4).

Example 3.5.1 (Generalized MAMDANI Case)

We assume $Q_0 = Q_1 = \dots = Q_n = \mathrm{Sup}$
$\kappa_0 = \kappa_1 = \dots = \kappa_n = \min$
$\alpha = \beta = \max$.

Then the assumptions 1, 2, 3, 2′ and 3′ are fulfilled, hence $FATI^{RB}_{\mathfrak{J}}$ and $FITA^{RB}_{\mathfrak{J}}$ are equivalent with respect to the set of all fuzzy sets on U.

Note that this result holds without any restrictions of the functions $\pi_1,\dots,\pi_n$ which interpret the rules $IF\,F_1\,THEN\,G_1,\dots,IF\,F_n\,THEN\,G_n$.

Acknowledgement

The authors wish to thank Ulrich Fieseler for useful scientific discussions on the subject and Mario Thalheim for his help in preparing the manuscript.

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